Compressive Sensing with Biorthogonal Wavelets via Structured Sparsity

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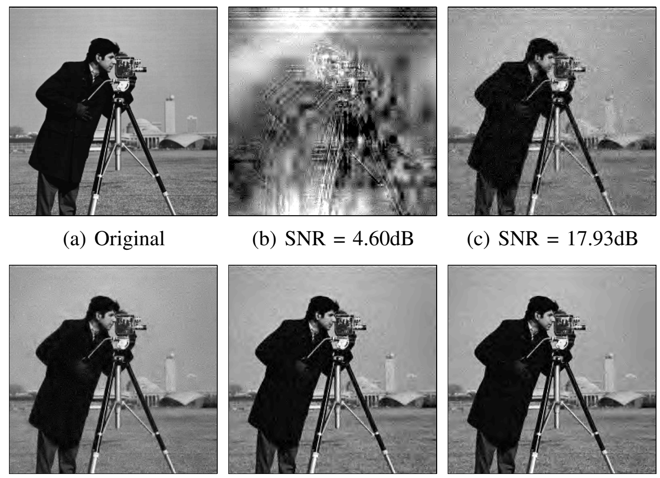
Compressive sensing (CS) merges the operations of data acquisition and compression by measuring sparse or compressible signals via a linear dimensionality reduction and then recovering them using a sparse-approximation based algorithm. A signal is K-sparse if its coefficients in some transform contain only K nonzero values; a signal is compressible if its coefficients decay rapidly when sorted by magnitude. The standard CS theory assumes that the sparsifying transform is an orthogonal basis.

Recently, progress has been made on CS recovery using more general, non-orthogonal transform based on *frames*. A tight frame consists of an analysis frame $\bar{\Psi}$ and a synthesis (dual) frame Ψ such that $\Psi\bar{\Psi}^T = I$. A signal x is analyzed by findings its transform coefficients via $\theta = \bar{\Psi}^T x$ and synthesized via $x = \Psi\theta$. Currently, provable CS recovery in a frame can be accomplished when either $(A1)$ the coherence of the frame (the maximum inner product between any two synthesis frame vectors) is low [1], or $(A2)$ the signal has a sparse or compressible analysis coefficient vector $\theta = \Psi^T x$ [2].

An important set of CS applications revolves around image acquisition, where CS has been used to boost the resolution of digital cameras at exotic wavelengths, reduce the scan time in MRI scanners, and so on. The sparsifying transforms of choice for image compression have long been the biorthogonal wavelet bases (BWBs), which are non-redundant tight frames with the property that the roles of the analysis and synthesis frames are interchangeable (i.e., $\Psi\bar{\Psi}^T = \Psi^T\bar{\Psi} = I$). In contrast to orthogonal wavelet bases (OWBs), BWBs can have symmetrical basis elements that induce less distortion on image edges when the coefficients θ are sparsified by thresholding. Symmetrical elements also yield more predictable coefficients, which boosts compression performance [3].

Unfortunately, BWBs not always satisfy condition $(A1)$. As an example, the CDF9/7 synthesis frame elements are far from orthogonal; indeed the coherence is slightly greater than $\frac{1}{2}$ for a 512×512 2-D synthesis frame. As a result, attempts at CS recovery using greedy techniques fails miserably (see Fig. 1(b)). In contrast, since the analysis and synthesis frames are interchangeable, then the approach in [2] is equivalent to standard ℓ_1-norm minimization, requiring $M = O(K \log(N/K))$ measurements.

We develop a new CS recovery technique for BWBs based on the notion of *structured sparsity* [4], which can provide near-optimal recovery from as few as $O(K)$ CS measurements. The particular model we apply is the quad-tree sparse/compressible model of [4], which is prevalent in BWB synthesis coefficient vectors for natural images. To provide recovery performance guarantees for signals with structured sparsity in a frame rather than a basis, we marry the concepts of the D-RIP [2], which requires near-isometry for signals with sparse synthesis coefficient vectors, with the structured RIP and RAmP [4] that restricts this near isometry only to signals with synthesis coefficient vectors that follow the quad-tree sparsity and compressibility models. The number of measurements needed in these cases is still $M = O(K)$. This class of signals includes the majority of the set of natural images, which can be shown to belong in a sufficiently smooth Besov space.



(a) Original (b) SNR = 4.60dB (c) SNR = 17.93dB

(d) SNR = 21.54dB (e) SNR = 22.14dB (f) SNR = 23.31dB

Fig. 1. (a) Original *Cameraman* image. Sparse recovery of the 512×512 *Cameraman* test image from $M = 60000$ noiselet measurements using: (b) CDF9/7 BWB and conventional CoSaMP [5] recovery; (c) D8 OWB and conventional CoSaMP; (d) CDF9/7 BWB and ℓ_1-norm minimization; (e) D8 OWB and tree-structured CoSaMP [4]; (f) CDF9/7 BWB and tree-structured CoSaMP. The CoSaMP-based algorithms use $K = 10000$.

The benefits of structured sparse recovery in a BWB are clear from Fig. 1(f), which boasts both a higher recovery signal-to-noise ratio (SNR) and noticably sharper edges and less ringing than the D8 OWB recovery in Fig. 1(c,e) or the CDF9/7 BWB recovery in FIg. 1(d). Our results can be easily extended to more general BWBs and redundant wavelet representations for smooth signals.

References

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