Compressive Sensing with Biorthogonal Wavelets via Structured Sparsity

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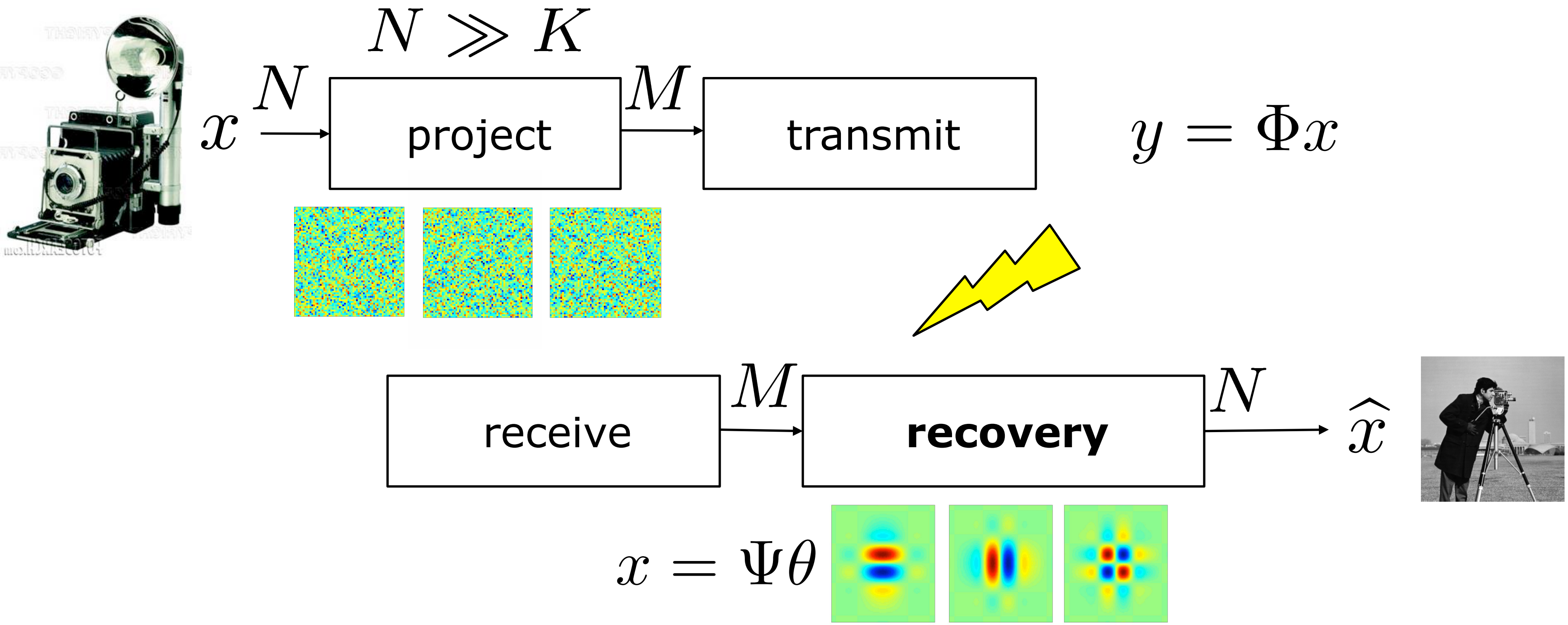
Compressive Imaging

- Data x K-sparse in ***orthonormal*** basis Ψ: $x = \Psi\theta$
- Measure linear projections onto ***incoherent*** basis Φ where data is ***not sparse***



- Reconstruct via ***optimization*** or ***greedy*** algorithms

$$\widehat{\theta} = \arg\min_{\theta \in \mathbb{R}^N} \|\theta\|_1 \text{ s.t. } \|y - \Phi\Psi\theta\|_2 \leq \epsilon$$

$$\widehat{x} = \Psi\widehat{\theta}$$

(CoSaMP, OMP, IHT, ...)

Restricted Isometry Property (RIP)

- Preserve distances between sparse/compressible signals

[Candès and Tao 2005]

- RIP of order $2K$ implies: for all K-sparse θ_1 and θ_2

$$(1-\delta)\|\theta_1-\theta_2\|_2^2 \leq \|\Phi\Psi(\theta_1-\theta_2)\|_2^2 \leq (1+\delta)\|\theta_1-\theta_2\|_2^2$$

RIP of order $2K$ enables K-sparse signal recovery

- **Random matrices** Φ lead to RIP with high probability if

$$M = O(K \log(N/K))$$

when Ψ is an orthonormal basis

- i.i.d. Subgaussian entries: Gaussian, Rademacher, ...

Example: Compressive Imaging

- Example: Recovery via ***CoSaMP***
  using ***2-D Daubechies-8*** wavelet

Original

SNR = 17.93dB

$$N = 262144,\ M = 60000$$

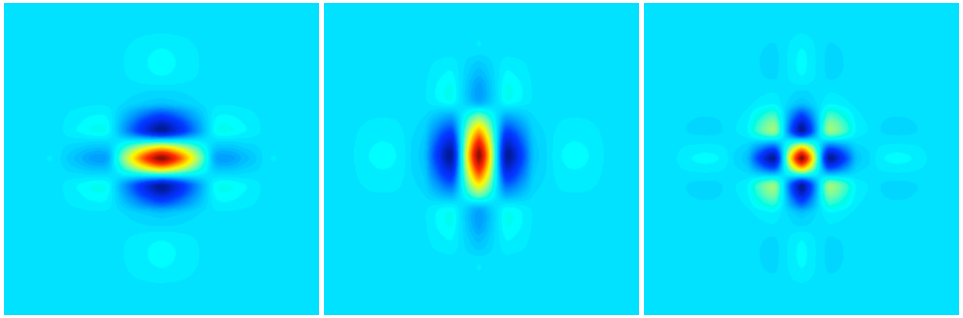
- 2-D wavelets and 2-D DCT are common

State-of-the-Art Image Compression

- **JPEG 2000**

JPEG JPEG 2000

- Lossy compression via ***transform coding***
- Cohen-Daubechies-Feauveau 9/7 Biorthogonal Wavelet ***(CDF 9/7)***



Properties of Biorthogonal Wavelets

- Biorthogonal wavelets involve:
  - an ***analysis*** basis $\overline{\Psi} = [\overline{\psi}_1 \ \overline{\psi}_2 \ \ldots \ \overline{\psi}_N]$
  - a ***synthesis*** basis $\Psi = [\psi_1 \ \psi_2 \ \ldots \ \psi_N]$

- Transform coding: $\theta = \overline{\Psi}^T x$, $x = \Psi\theta$ $\longrightarrow$ $\Psi\overline{\Psi}^T = \mathbf{I}$

- Analysis and synthesis bases are ***not orthonormal***: $\overline{\Psi}\,\overline{\Psi}^T \neq \mathbf{I}$ $\quad \Psi\Psi^T \neq \mathbf{I}$

- Standard guarantees and algorithms are not necessarily ***suitable*** for biorthogonal wavelets

Beyond Orthonormal Bases

- ***Dictionary***: arbitrary matrix (basis/frame) that provides sparsity
- Dictionary ***coherence***

$$\mu(\Psi) = \arg \min_{1 \leq i \neq j \leq N} |\langle \psi_i, \psi_j \rangle|$$

- ***Theorem***: [Rauhut, Schnass, Vandergheynst 2008]

  If Φ has $M = O(K \log(N/K))$ rows and

$$\mu(\Psi) \leq \frac{1}{17(K-1)}$$

  then the matrix $\Upsilon = \Phi\Psi$ has RIP of order K

- Sadly, CDF 9/7 synthesis basis has ***large coherence:*** $\mu(\Psi) > 0.5$

CS with Coherent Dictionaries

- For tight frame dictionaries with arbitrary coherence, can use ℓ_1-***analysis***:
  ℓ_1-norm minimization for analysis coefficients

$$\widehat{x} = \arg\min_{x \in \mathbb{R}^n} \|\overline{\Psi}^T x\|_1 \text{ s.t. } \|y - \Phi x\|_2 \leq \epsilon$$

- New Ψ-***RIP*** preserves distances between ***signal vectors*** instead of coefficient vectors
- Ψ-RIP is tailored to dictionary Ψ and **enables guarantees** for recovery via ℓ_1-analysis (in contrast to ℓ_1-synthesis)
- Random matrix has Ψ-RIP if $M = O(K \log(N/K))$

[Candès, Eldar, Needell, Randall 2010]

Compressive Imaging via Biorthogonal Wavelets

- Example: Recovery via ***CoSaMP*** using ***CDF 9/7*** wavelet

Original

SNR = 4.6dB

$$N = 262144, \; M = 60000$$

Compressive Imaging via Biorthogonal Wavelets

- Example: Recovery via ℓ_1-***synthesis***
  using ***CDF 9/7*** wavelet

Original

SNR = 21.54dB!

$$N = 262144, \; M = 60000$$

Benefits of Biorthogonal Wavelets

- Why does ℓ_1-synthesis work well?
- Because biorthogonal wavelet analysis and synthesis bases are "***interchangeable***":

$$\Psi\overline{\Psi}^T = \overline{\Psi}^T\Psi = \mathbf{I}$$

- Thus, we have $\overline{\Psi}^T x = \overline{\Psi}^T \Psi\theta = \theta$ and the following two formulations are ***equivalent***:

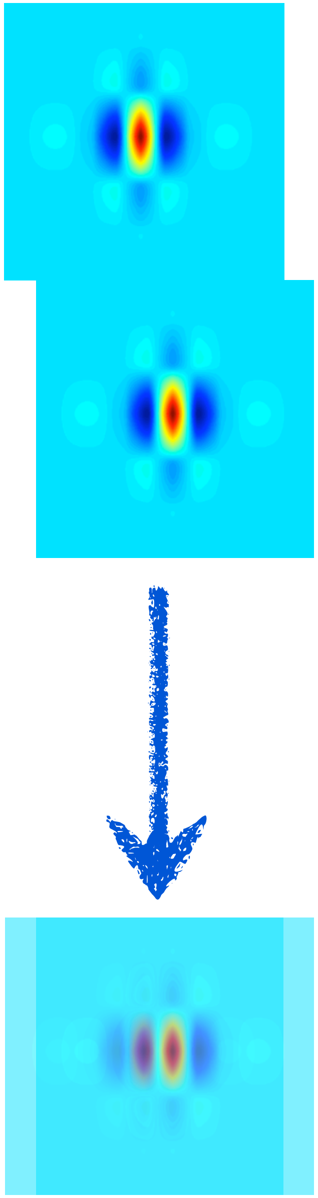
$$\widehat{\theta} = \arg\min_{\theta \in \mathbb{R}^N} \|\theta\|_1 \text{ s.t. } \|y - \Phi\Psi\theta\|_2 \leq \epsilon$$

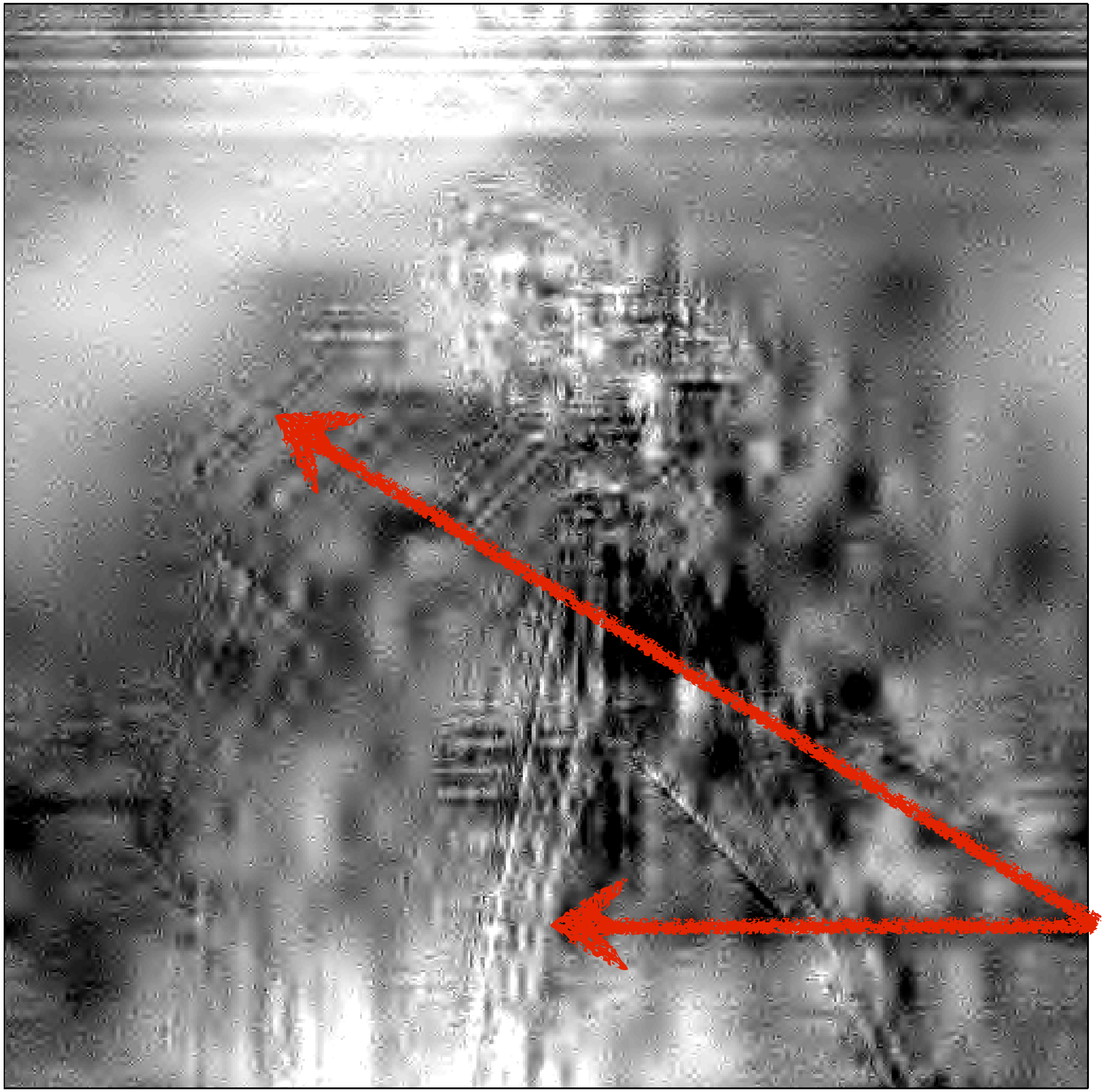
(ℓ_1-synthesis)

$$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \|\overline{\Psi}^T x\|_1 \text{ s.t. } \|y - \Phi x\|_2 \leq \epsilon$$

(ℓ_1-analysis)

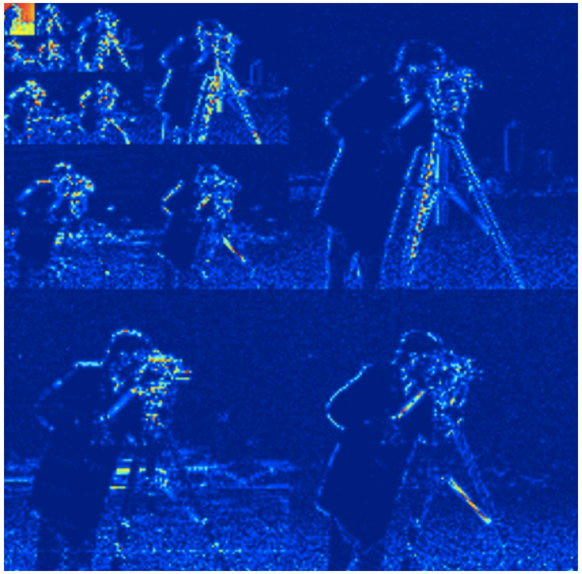
CDF 9/7 Recovery Artifacts

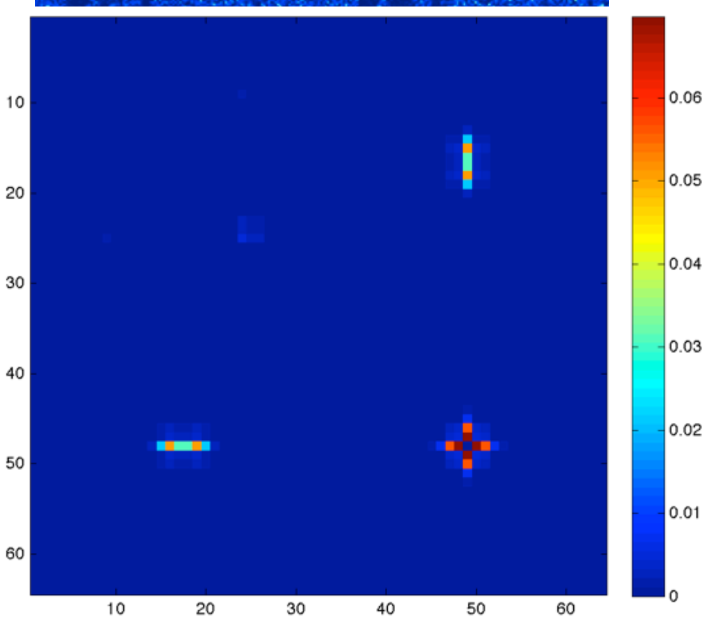




“Ringing”

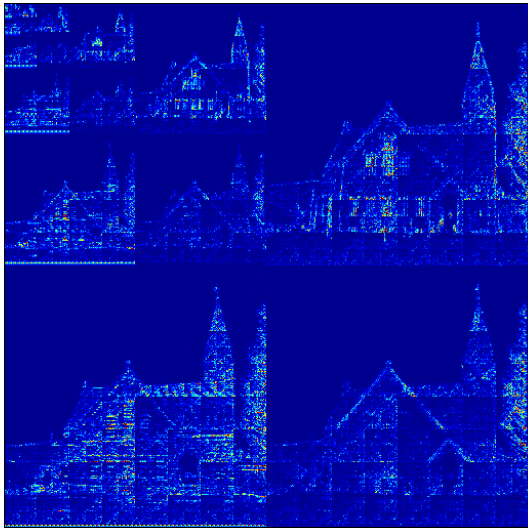
Coherence of Biorthogonal Wavelets

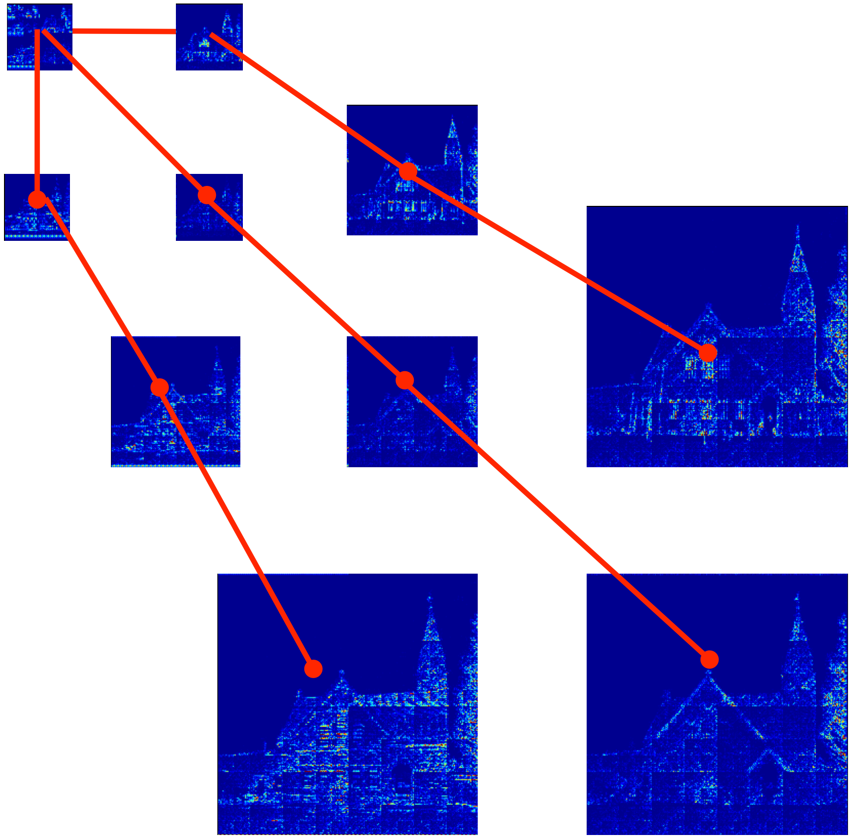




- Each wavelet is coherent with ***spatial neighbors*** across different wavelet orientations and scales
- Ringing artifacts caused by ***ambiguity*** due to coherent/neighbor wavelets during sparse wavelet selection
- Can ***inhibit*** supports that include coherent pairs of neighboring wavelets

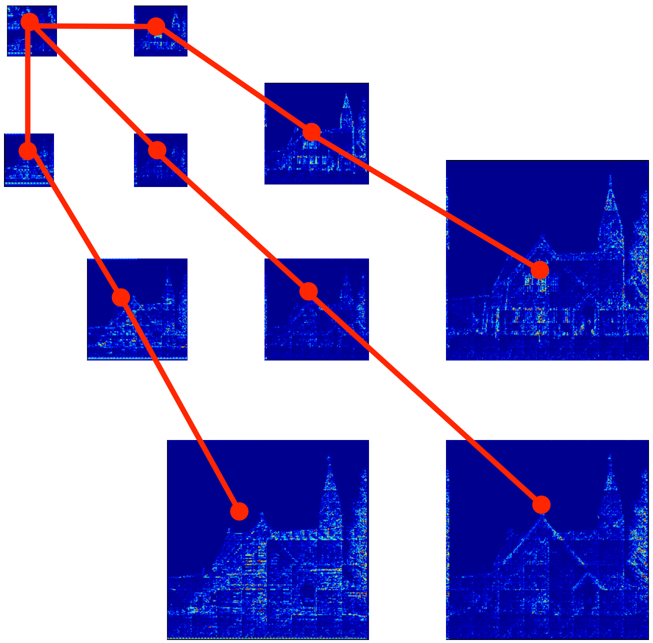
Structured Sparsity Models



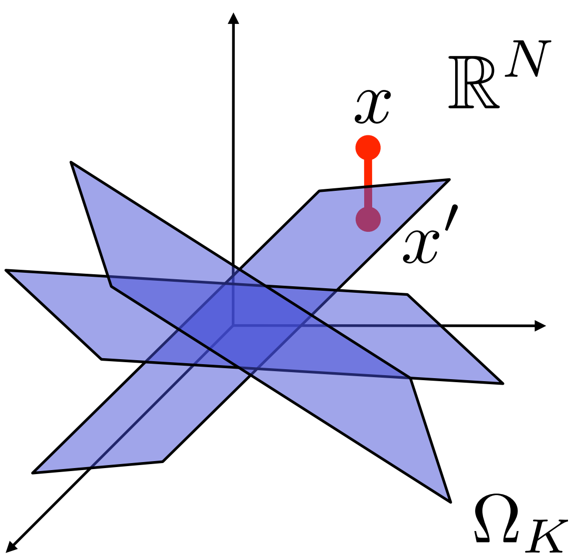


- ***Promote*** structure common in natural images
- ***Inhibit*** selection of additional highly-coherent wavelet pairs

Structured Sparse Recovery Algorithms



- Modify existing greedy algorithms that rely on thresholding (e.g. CoSaMP)
- Replace thresholding with **best structured sparse approximation** that finds the closest point to input x in a *restricted union of subspaces* that encodes structure:

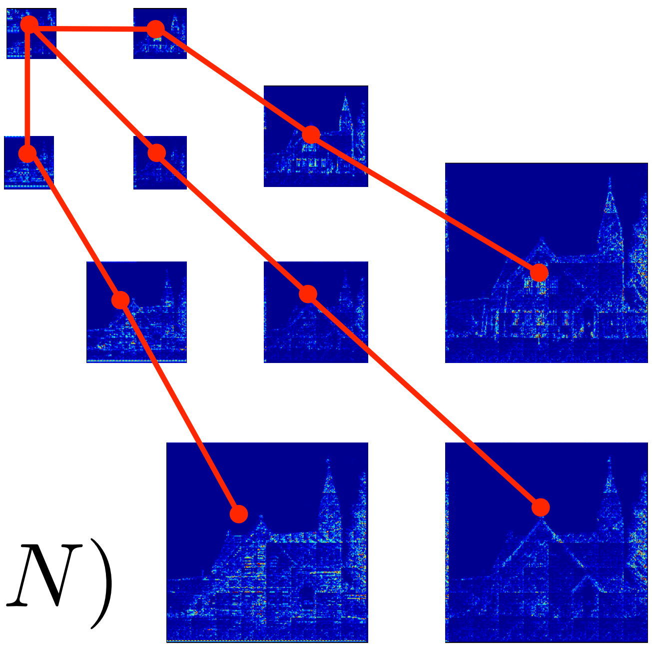


$$x' = \arg\min_{\overline{x} \in \Omega_K} \|x - \overline{x}\|_2$$

[Baraniuk, Cevher, Duarte, Hegde 2010]

Connected Rooted Subtree Sparsity

- **Structure:** K-sparse coefficients
  + nonzero coefficients lie on a rooted subtree



- **Approximation algorithm:**
– condensing sort and select [Baraniuk]
– dynamic programming [Donoho]
– computational complexity: $\mathcal{O}(N \log N)$

- **Measurements needed:**
  for random matrices with i.i.d entries,

without structure

$$M = \mathcal{O}\left(K \log \frac{N}{K}\right)$$

with structure

$$M = \mathcal{O}(K)$$

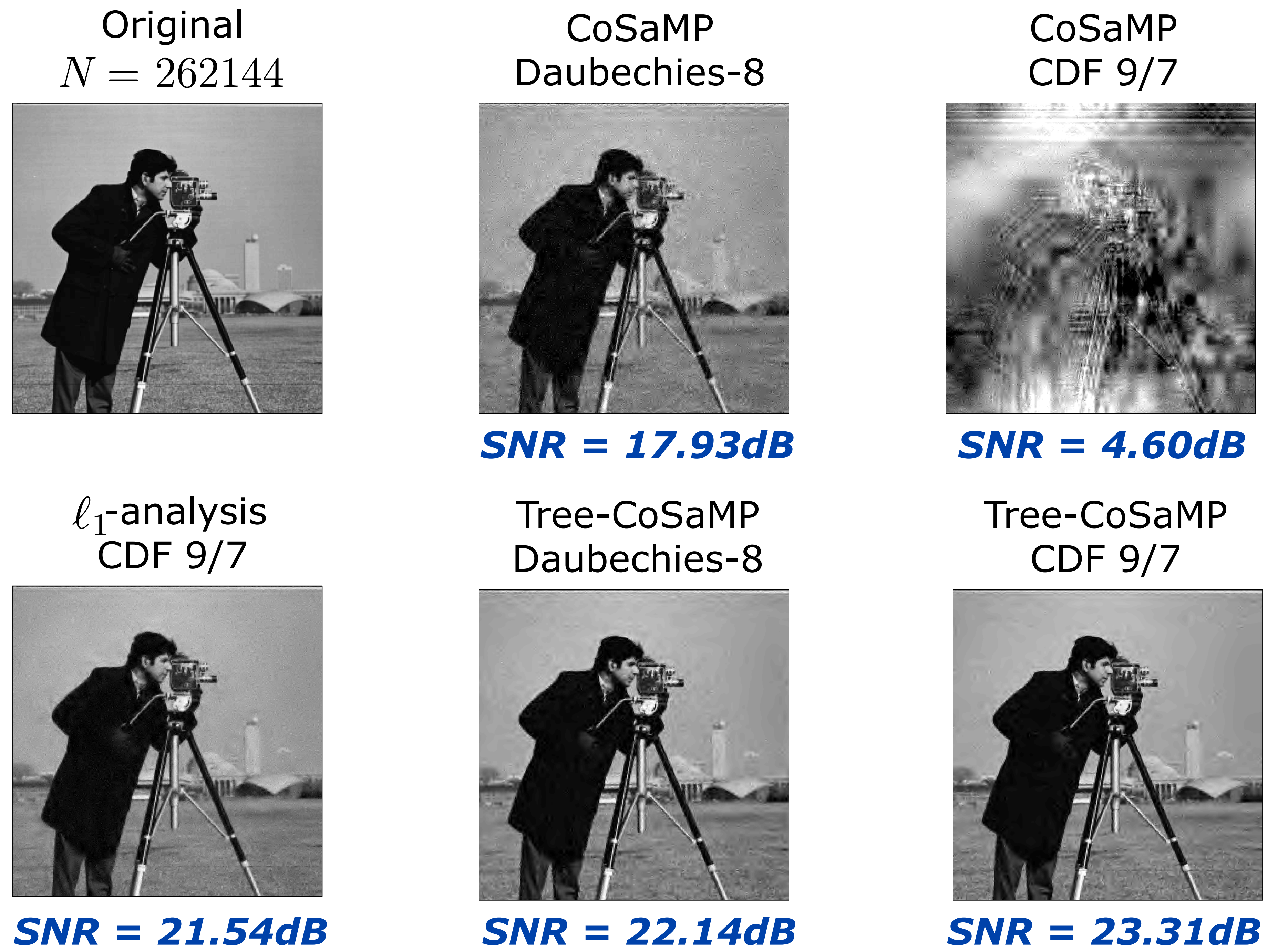
CS via Biorthogonal Wavelets

- Example: Recovery via ***tree-CoSaMP***
  using ***CDF 9/7*** wavelet

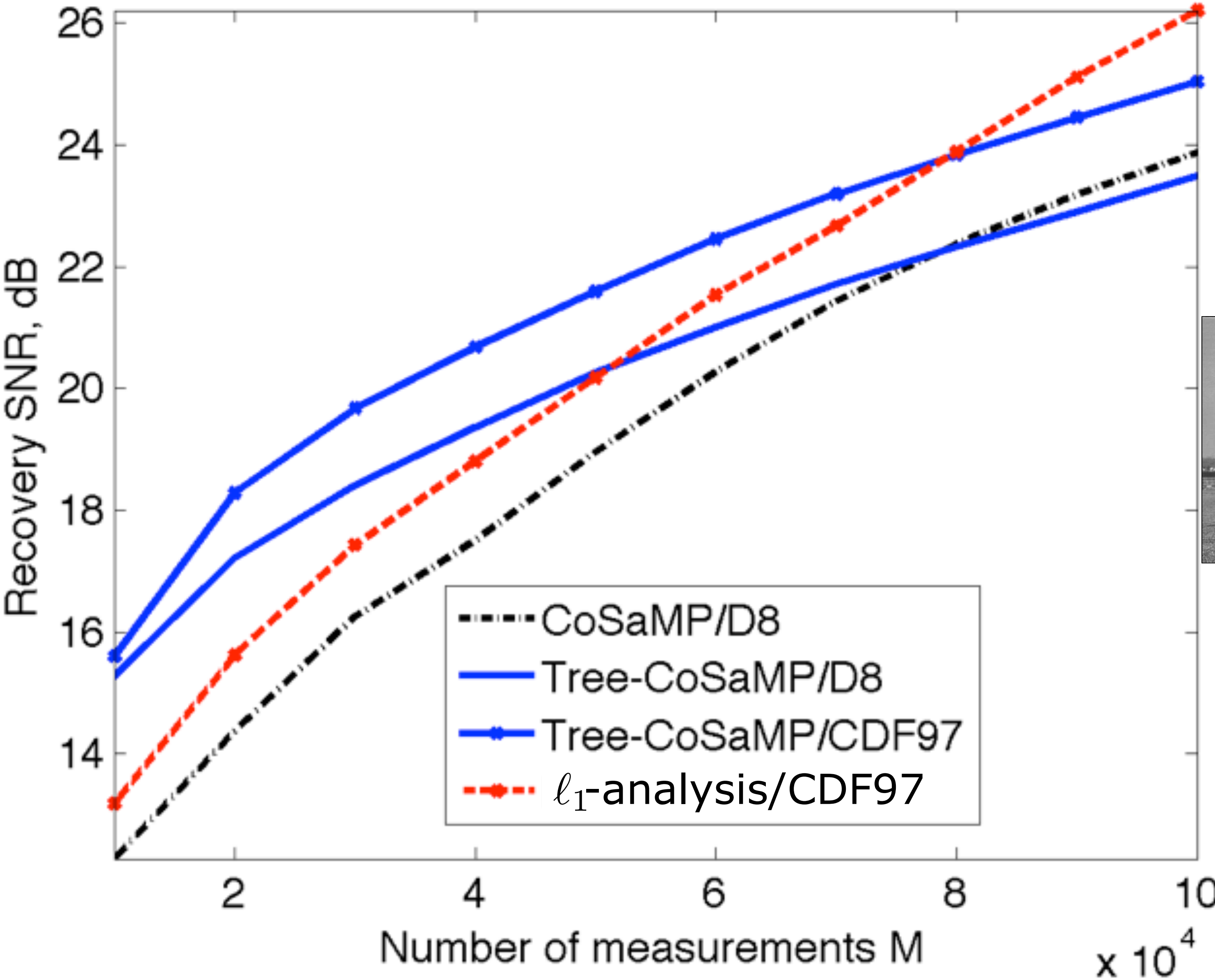
Original

SNR = 23.31dB

$$N = 262144,\ M = 60000$$


Original
$N = 262144$
CoSaMP
Daubechies-8
SNR = 17.93dB
CoSaMP
CDF 9/7
SNR = 4.60dB
ℓ_1-analysis
CDF 9/7
SNR = 21.54dB
Tree-CoSaMP
Daubechies-8
SNR = 22.14dB
Tree-CoSaMP
CDF 9/7
SNR = 23.31dB

Performance Comparison - Cameraman



$N = 262144$

Conclusions

- ***Structured sparsity*** enables improved greedy algorithms for compressive imaging with ***2-D biorthogonal wavelets***
  - ***promote*** structure present in wavelet representations of natural images
  - ***inhibit*** interference from neighboring wavelets that do not match model
  - ***simple-to-implement*** modifications to recovery that are ***computationally efficient***
  - ***reduced*** number of random measurements required for ***improved*** image recovery
- Current and future work:
  - analytical study of 2-D biorthogonal wavelets (coherence, RIP) for compressive imaging
  - Extensions to redundant wavelet frames

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