Kronecker Product Matrices for Compressive Sensing

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Compressed Sensing: Using signal structure, dimensionality reduction, approximation algorithms to acquire data @ sampling rate below Nyquist

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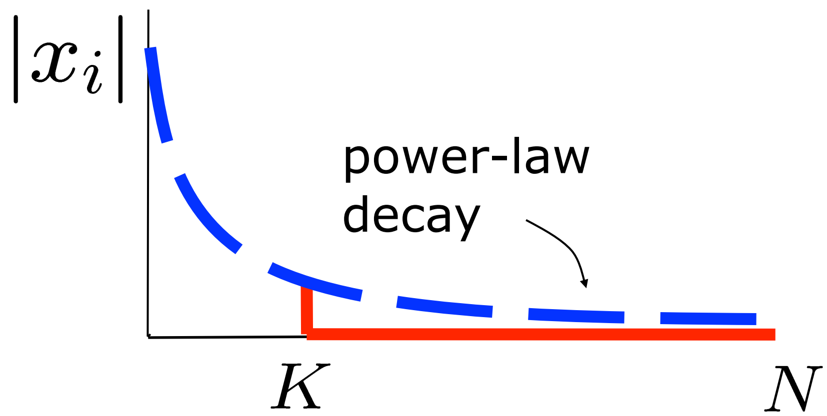
Real-World Signals

- **Compressible** signal: sorted coordinates decay rapidly to zero

  well-approximated by a K-sparse signal
  (by thresholding or sparse approx.)

  - model: weak ℓ_p-ball: $|x_i| < S i^{-1/p}$

$$\sigma_K(x) := \|x - x_K\|_2 \leq (ps)^{-1/2} S K^{-s}$$



$$s = \frac{1}{p} - \frac{1}{2}$$

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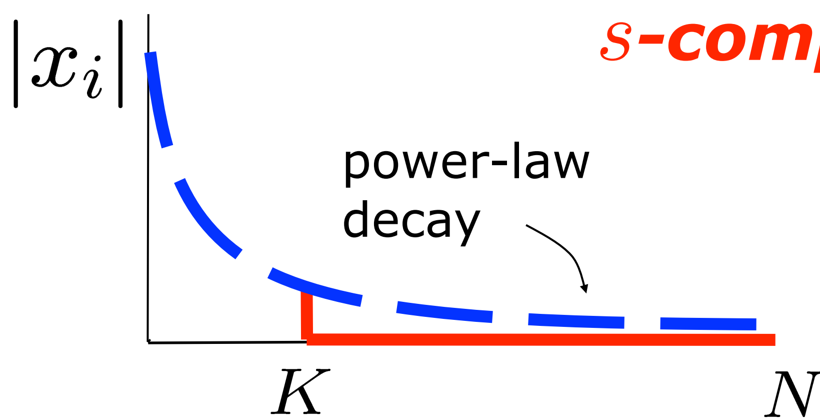
s-compressible



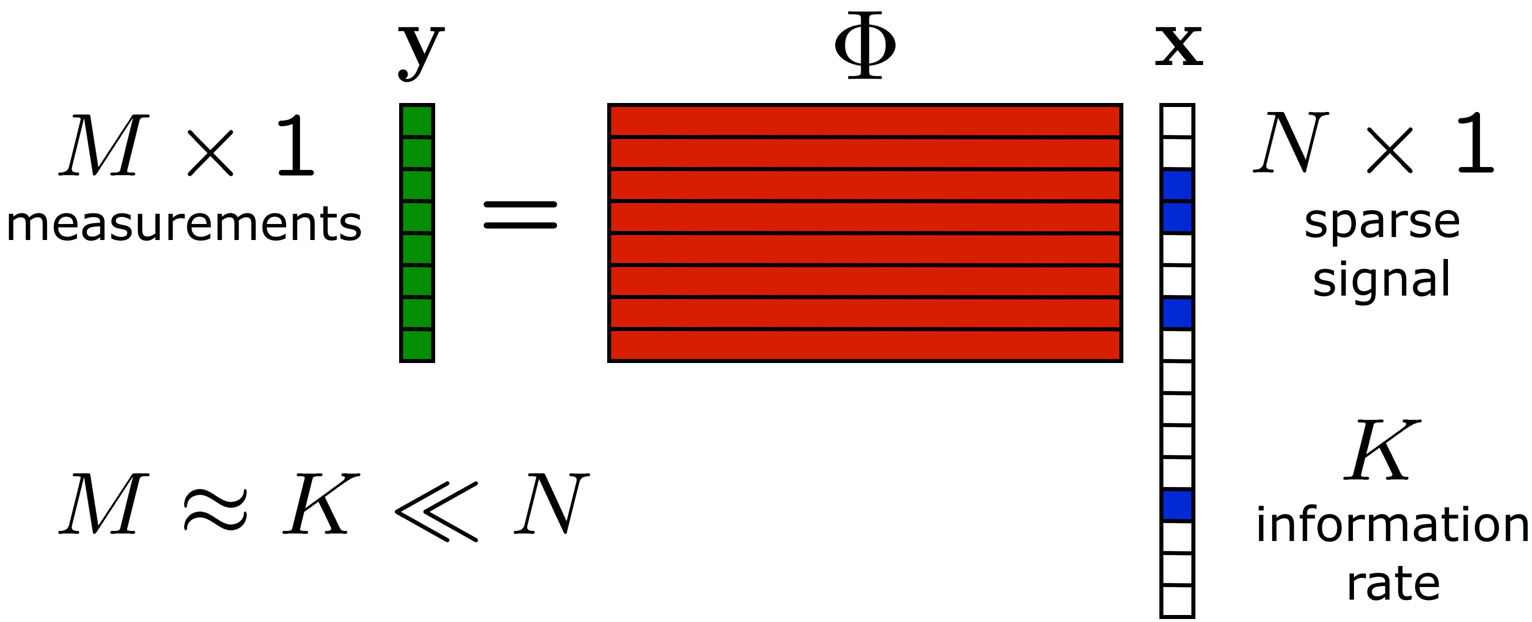
$$s = \frac{1}{p} - \frac{1}{2}$$

From Samples to *Measurements*

- Replace **samples** by more general **encoder** based on a few linear projections (inner products)

- Restricted Isometry Property - random matrices

$\mathbf{y} = \Phi\mathbf{x}$, $\mathbf{x}$ is sparse



$$M \approx K \ll N$$

Mutual Coherence

- Measurements selected from **preset basis** Φ
  (ex: 2D-Fourier measurements in MRI,
  permuted Walsh measurements in
  single-pixel camera)

- **Mutual coherence** between *sparsity basis* Ψ and *measurement basis* Φ:

$$\mu(\Phi, \Psi) = \max_{\phi, \psi} \left| \langle \phi, \psi \rangle \right|$$

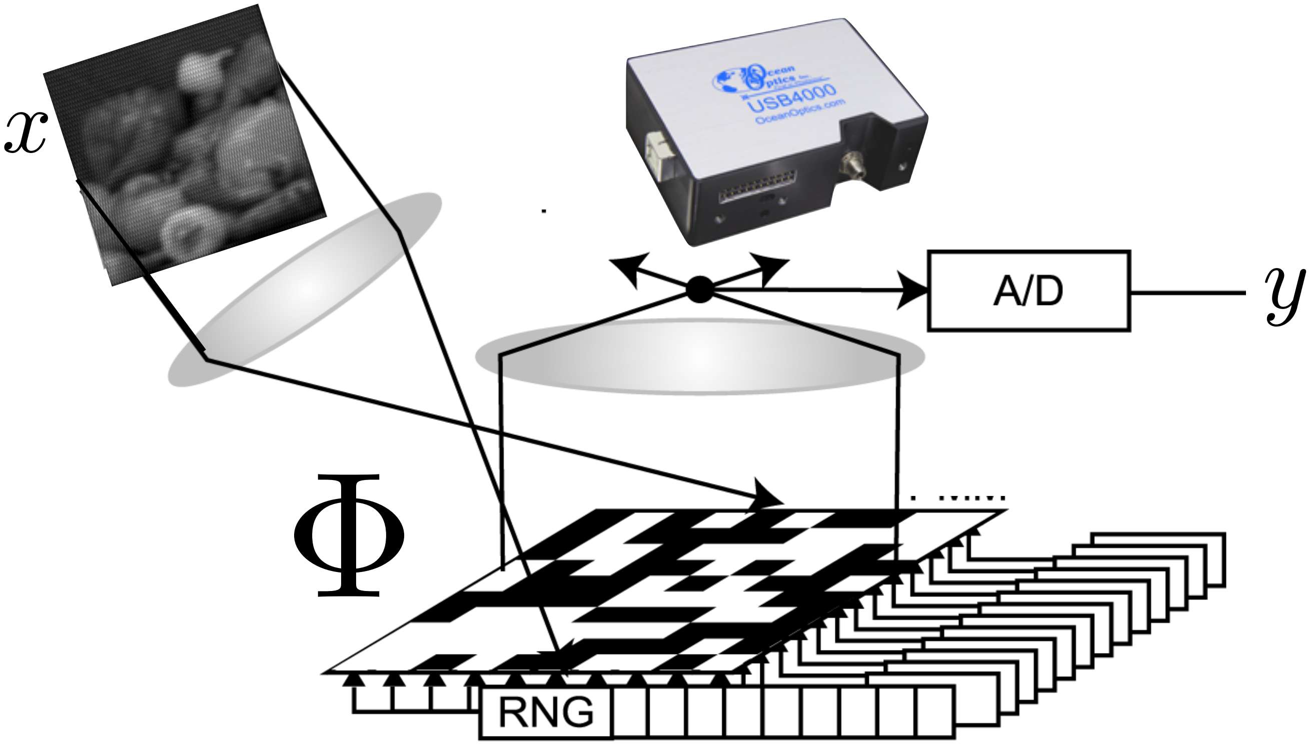
- Number of measurements needed for recovery:

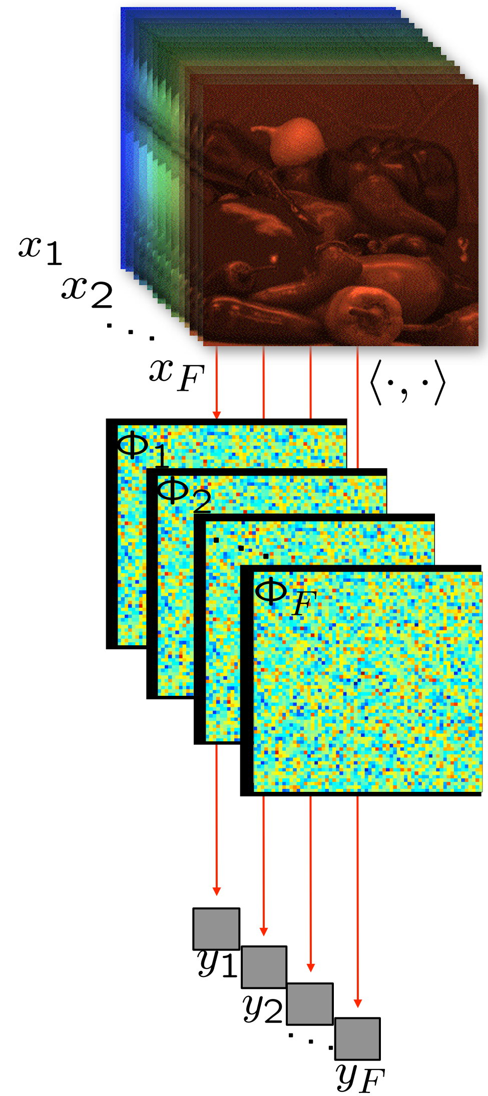
$$M \geq C K N \mu^2(\Phi, \Psi) \log N$$

$$\mu(\Phi, \Psi) \in \left[\frac{1}{\sqrt{N}}, 1 \right]$$

[Candès and Romberg]

Single Pixel Hyperspectral Imaging





Hyperspectral datacube

- ***Multidimensional*** signals
- ***Distributed*** measurements

[with D. Takhar, J. Laska, T. Sun, K. Kelly, R. Baraniuk]

Distributed
Compressive Sensing (DCS)

Distributed Sensing

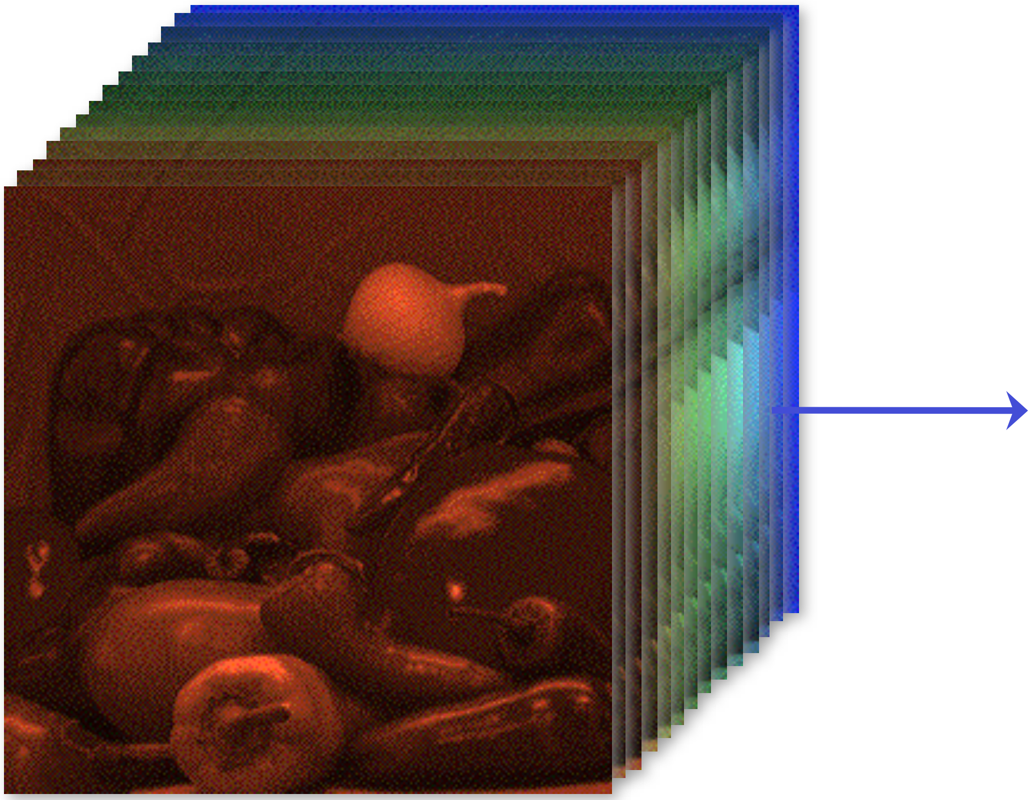
$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_J \end{bmatrix} = \begin{bmatrix} \Phi_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Phi_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Phi_J \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_J \end{bmatrix}$$

$$\mathbf{Y} \qquad \Phi \qquad \mathbf{X}$$

Joint Recovery

[Joint work with D. Baron, M. Wakin, S. Sarvotham, and R. Baraniuk]

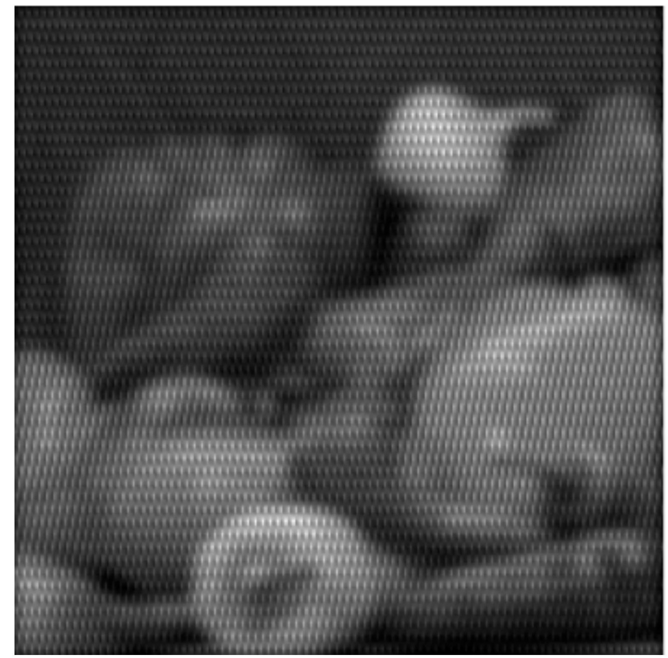
Inspiration: Hyperspectral Imaging



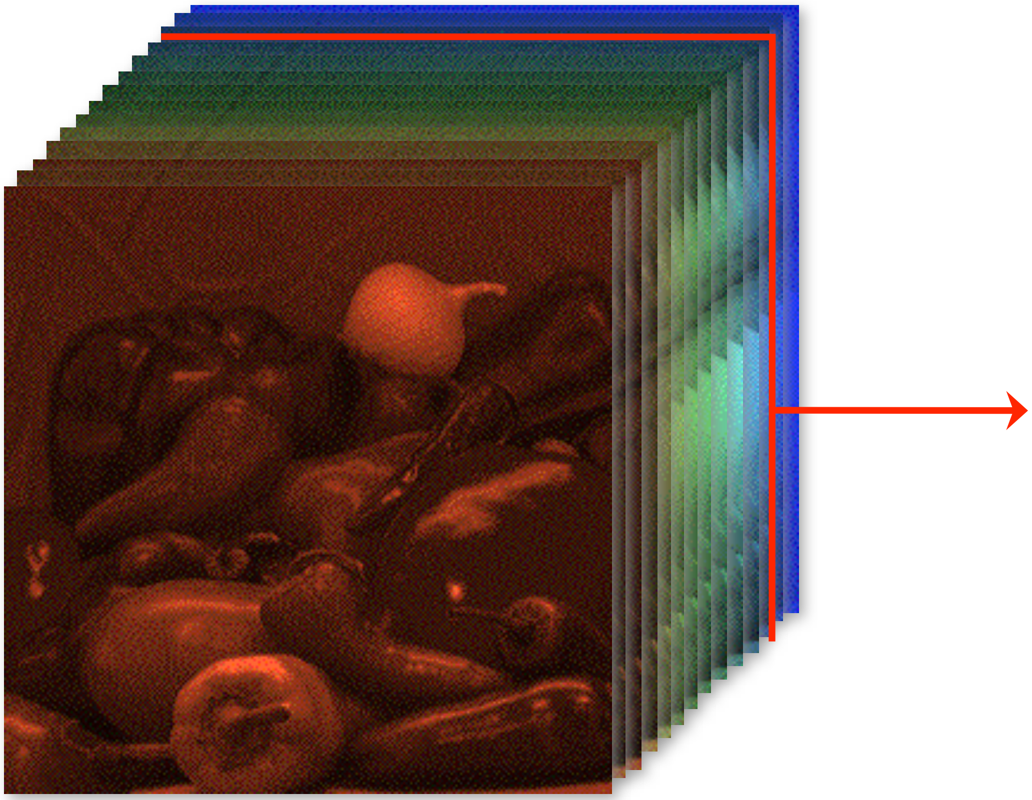
$\mathbf{X}$

Hyperspectral datacube

Each band
measured separately



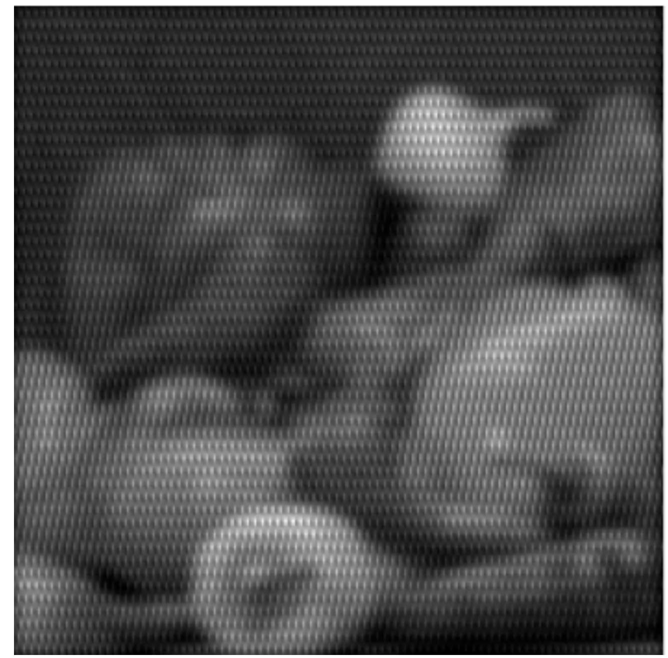
Inspiration: Hyperspectral Imaging



$\mathbf{X}$

Hyperspectral datacube

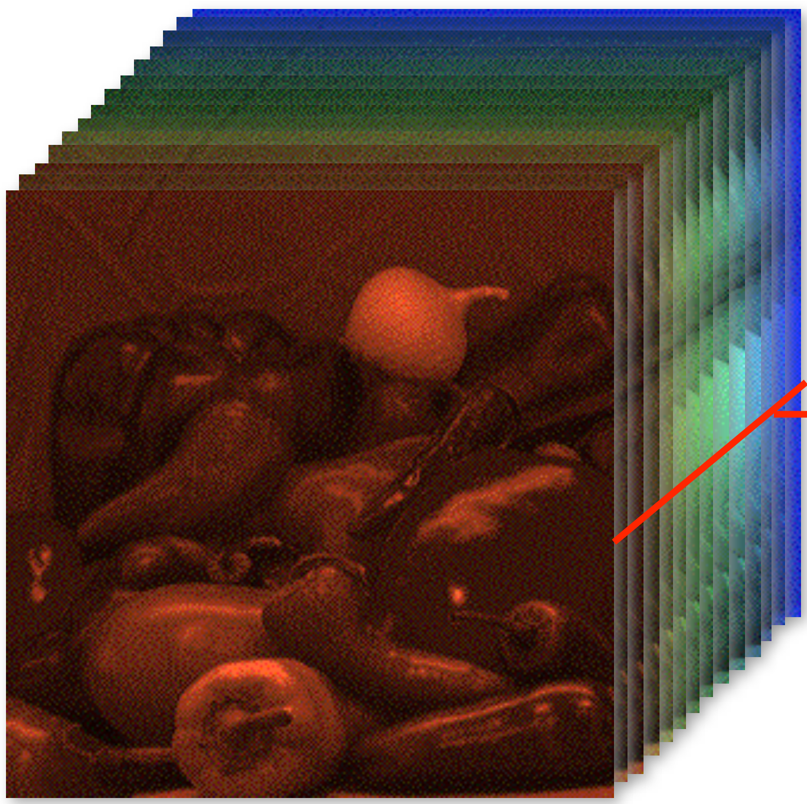
Intra-signal correlations:



spatial sparsity (wavelets)

Ψ_S

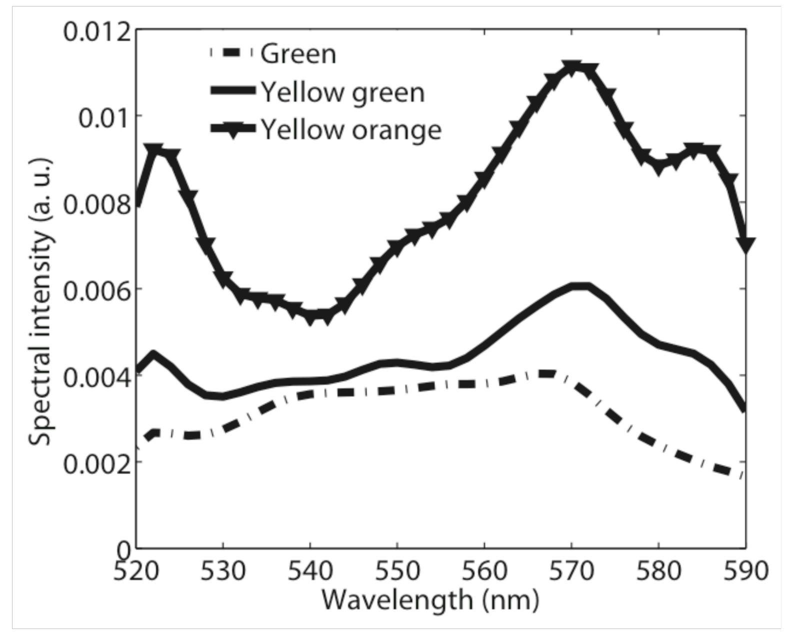
Inspiration: Hyperspectral Imaging



$\mathbf{X}$

Hyperspectral datacube

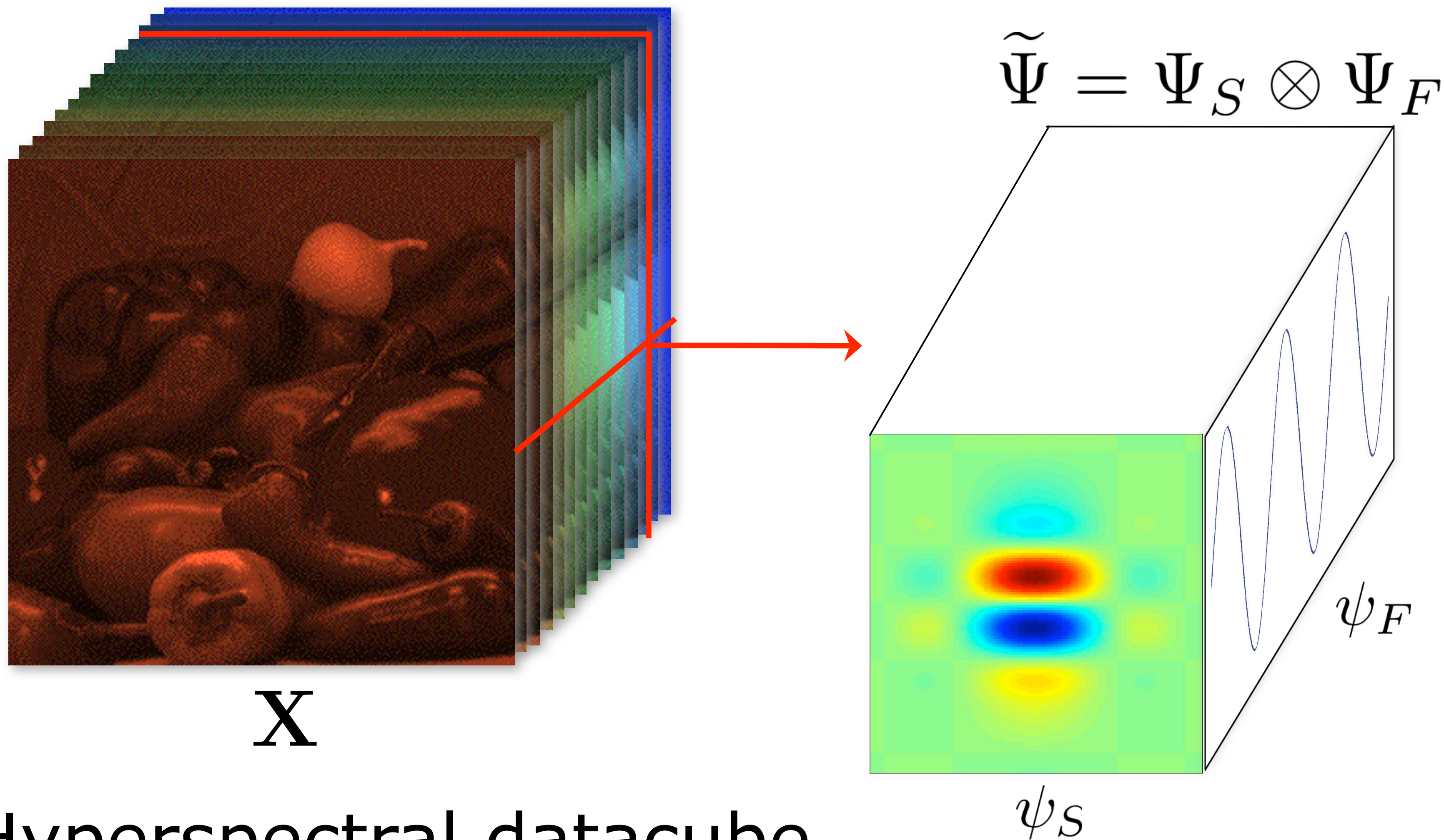
Inter-signal correlations:



spectral sparsity (Fourier)

Ψ_F

Idea: Kronecker Products



Hyperspectral datacube

One More Thing...

$$\Phi = \begin{bmatrix} \Phi_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Phi_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Phi_J \end{bmatrix}$$

One More Thing...

$$\widetilde{\Phi} = \begin{bmatrix} \Phi & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Phi & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Phi \end{bmatrix}$$

$$\widetilde{\Phi} = \mathbf{I} \bigotimes \Phi$$

Kronecker Products for DCS

$$\widetilde{\Phi} = \mathbf{I} \otimes \Phi \qquad \widetilde{\Psi} = \Psi_1 \otimes \Psi_2$$

Kronecker Compressive Sensing

$$\widetilde{\Phi} = \Phi_1 \otimes \Phi_2 \qquad \widetilde{\Psi} = \Psi_1 \otimes \Psi_2$$

CS performance metrics
for
Kronecker product matrices

Signal classes that are
sparse/compressible
in Kronecker product bases

Kronecker Compressive Sensing

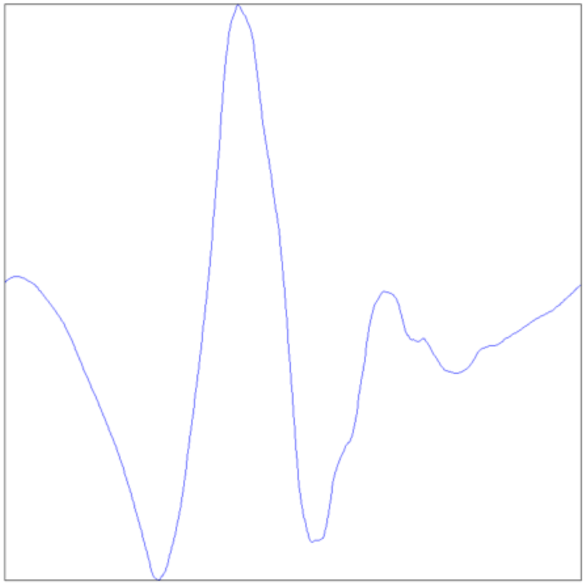
$$\widetilde{\Phi} = \Phi_1 \otimes \Phi_2 \qquad \widetilde{\Psi} = \Psi_1 \otimes \Psi_2$$

CS performance metrics
for
Kronecker product matrices

- ***Mutual Coherence***:
  For matrices $\Phi_j, \ \Psi_j, \ 1 \leq j \leq J,$

$$\mu(\Phi_1 \otimes \ldots \otimes \Phi_J, \Psi_1 \otimes \ldots \otimes \Psi_J) = \prod_{j=1}^{J} \mu(\Phi_j, \Psi_j)$$

Example: Wavelets



ν : scaling function
ψ : wavelet function
$\psi_{i,j}$: translated and dilated wavelet

$$\psi_{i,j}(t) = \frac{1}{2^{i/2}}\psi\left(\frac{t}{2^i} - j\right)$$

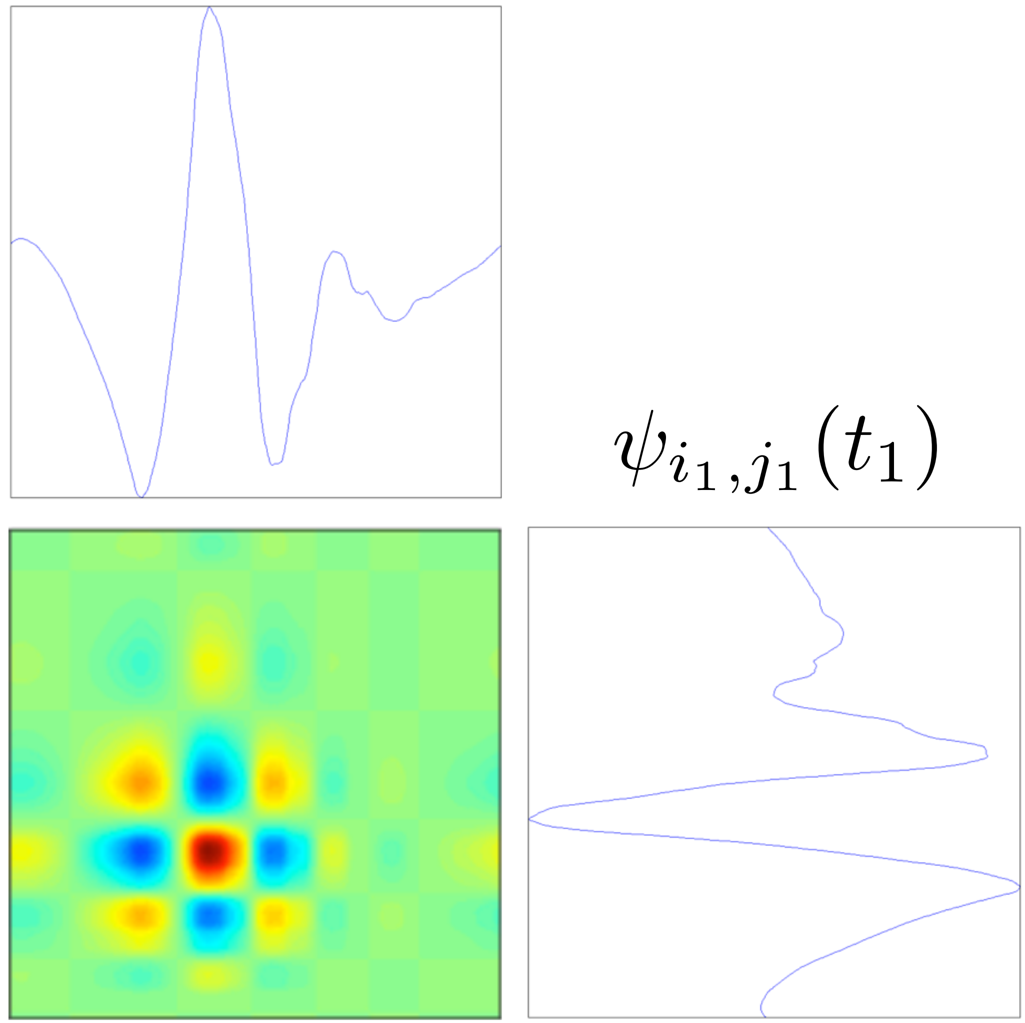
$$g = v_0\nu + \sum_{i\geq 0}\sum_{j=0}^{2^i-1} w_{i,j}\psi_{i,j},$$

$$\begin{aligned} v_0 &= \langle g, \nu\rangle \\ w_{i,j} &= \langle g, \psi_{i,j}\rangle \end{aligned}$$

Higher-Dimensional Wavelets

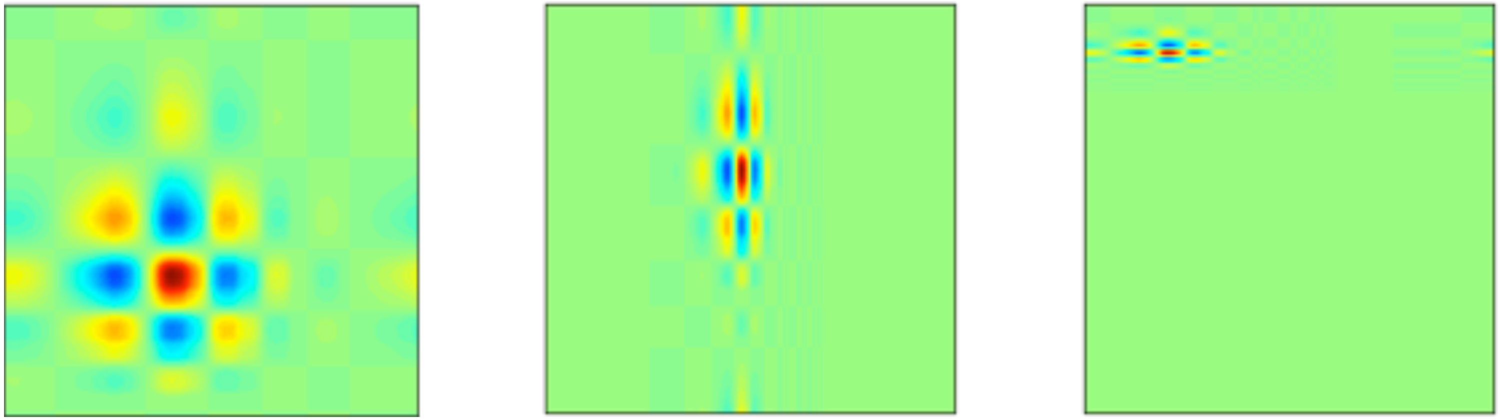
$$\psi_{i_2,j_2}(t_2)$$

$$\psi_{i_1,j_1}(t_1)$$

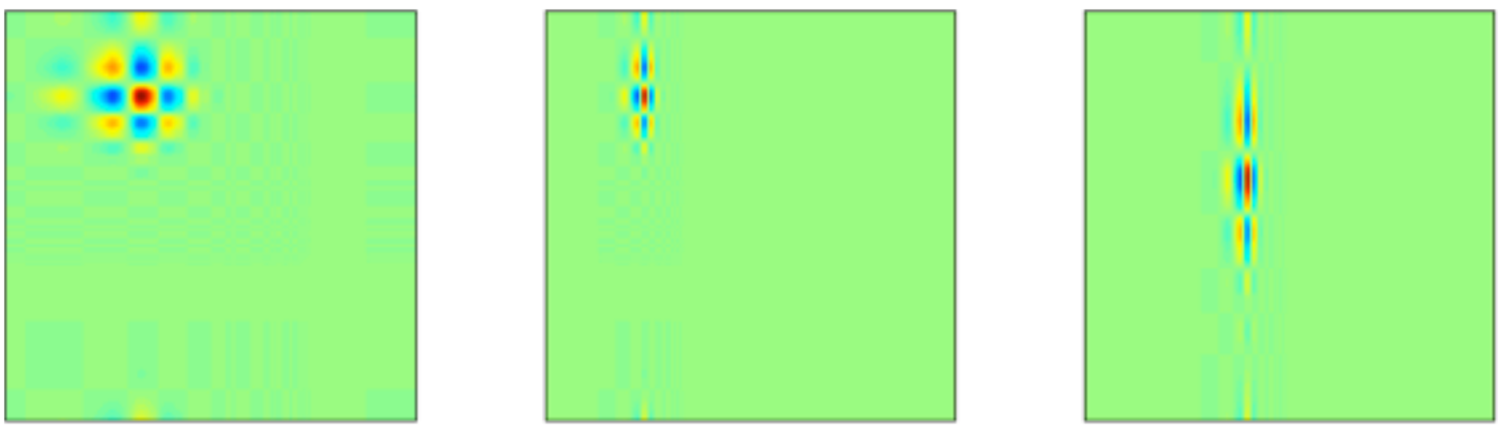


$$\psi_{i_1,j_1,i_2,j_2}(t_1, t_2) = \psi_{i_1,j_1}(t_1) \otimes \psi_{i_2,j_2}(t_2)$$

Higher-Dimensional Wavelets



$$\psi_{i_1,j_1,i_2,j_2}(t_1,t_2) = \psi_{i_1,j_1}(t_1) \otimes \psi_{i_2,j_2}(t_2)$$



Isotropic	Anisotropic	Hyperbolic
$i_1 = i_2$	$i_1 a_1 = i_2 a_2$	i_1 ? i_2

Compressibility in Wavelet Bases

- ***Isotropic Besov Space*** $B^s_{p,q}$***:***
  signals in L_p with s degrees of smoothness (derivatives) in all directions

- Isotropic wavelet characterization:

$$|g|_{B^s_{p,q}} \asymp \left(\sum_i 2^{iqs} \left(\sum_j |w_{i,j}|^p \right)^{q/p} \right)^{1/q}$$

- Compressibility: If the wavelet ψ has more than s vanishing moments, then a signal $g \in B^s_{p,q}$ is s***-compressible*** in an isotropic wavelet basis

Compressibility in Wavelet Bases

- ***Anisotropic Besov Space*** $B_{p,q}^{\overline{s}}$, $\overline{s} = (s_1, \ldots, s_D)$**:** D-dimensional signals in L_p with s_d degrees of smoothness (derivatives) in d^{th} dimension, $1 \leq d \leq D$
- Anisotropic wavelet characterization: anisotropy $a_1, \ldots, a_D$

$$|g|_{B_{p,q}^{\overline{s}}} \asymp \left(\sum_{n \in \mathbb{N}} 2^{n a_1 s + r(n)(1/D - 1/p)} \left(\sum_{j_1, \ldots, j_D} |w_{r_1(n), j_1, \ldots, r_D(n), j_D}|^p \right)^{q/p} \right)^{1/q}$$

$$r_d(n) = \left\lfloor \frac{a_i n}{a_1} \right\rfloor, \ 1 \leq d \leq D \qquad r(n) = \sum_{d=1}^{D} r_d(n)$$

$$s = \frac{D}{\sum_{d=1}^{D} \frac{1}{s_d}}$$

Compressibility in Wavelet Bases

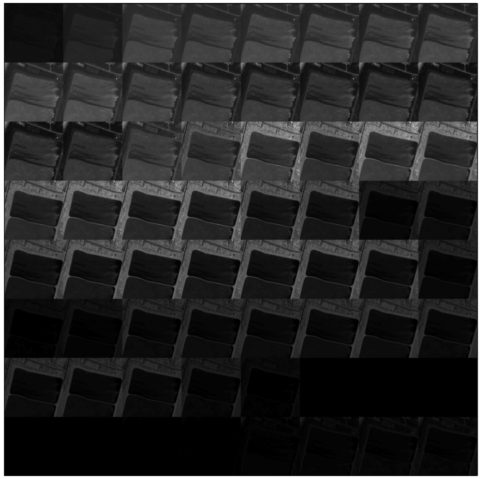
- If the wavelet ψ is sufficiently smooth*, then a signal $g \in B^{\overline{s}}_{p,q}$ is ***s-compressible*** in an ***anisotropic*** wavelet basis with $a_d = s/s_d,\ 1 \leq d \leq D$

- If the wavelet ψ is sufficiently smooth*, then a signal $g \in B^{\overline{s}}_{p,q}$ is ***s'-compressible*** in an ***isotropic*** wavelet basis with $s' = \min_{1 \leq d \leq D} s_d$

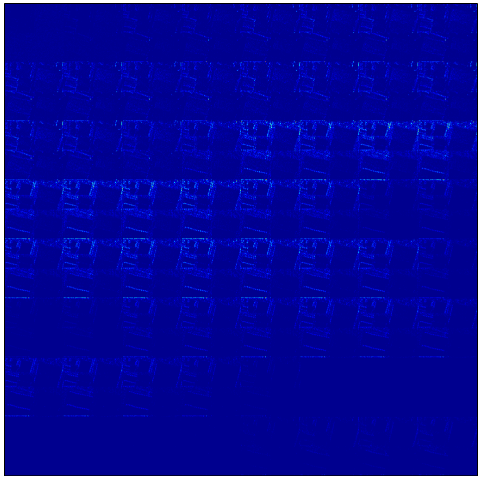
- If the wavelet ψ is sufficiently smooth*, then a signal $g \in B^{\overline{s}}_{p,q}$ is ***s-compressible*** in a ***hyperbolic*** wavelet basis $\widetilde{\Psi} = \Psi \otimes \ldots \otimes \Psi$

$$s = \frac{D}{\sum_{d=1}^{D} \frac{1}{s_d}}$$

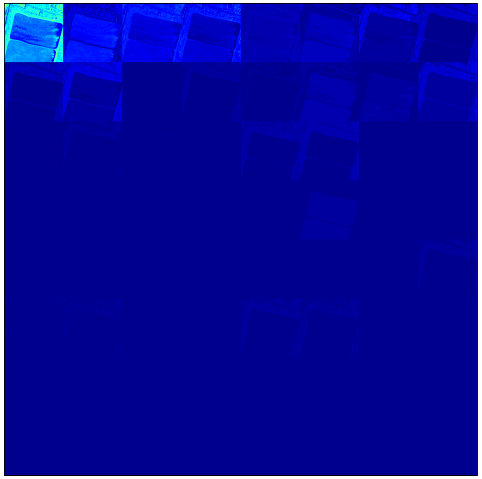
Example: Hyperspectral Data (AVIRIS)



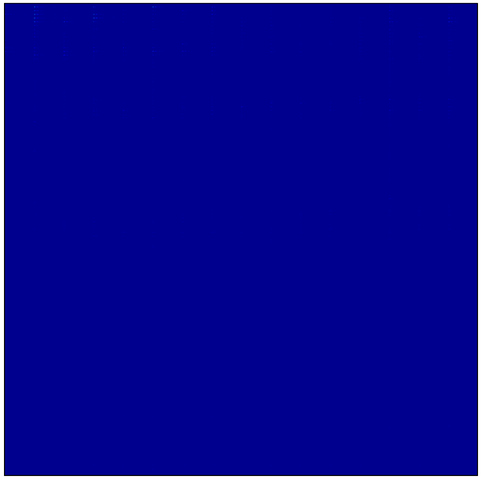
Original data



Spatial Wavelets



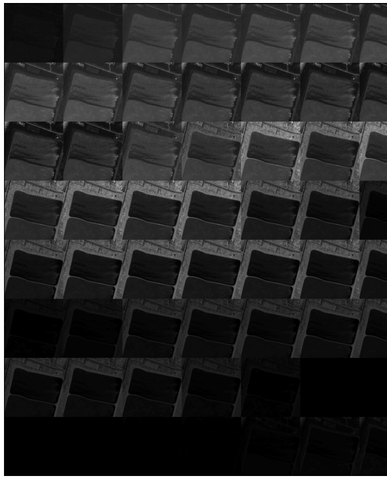
Spectral Wavelets



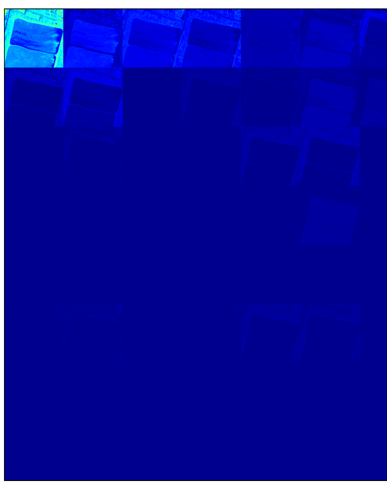
Hyperbolic Wavelets

$$\mathbf{x} \in \mathbb{R}^{128\times128\times64}$$

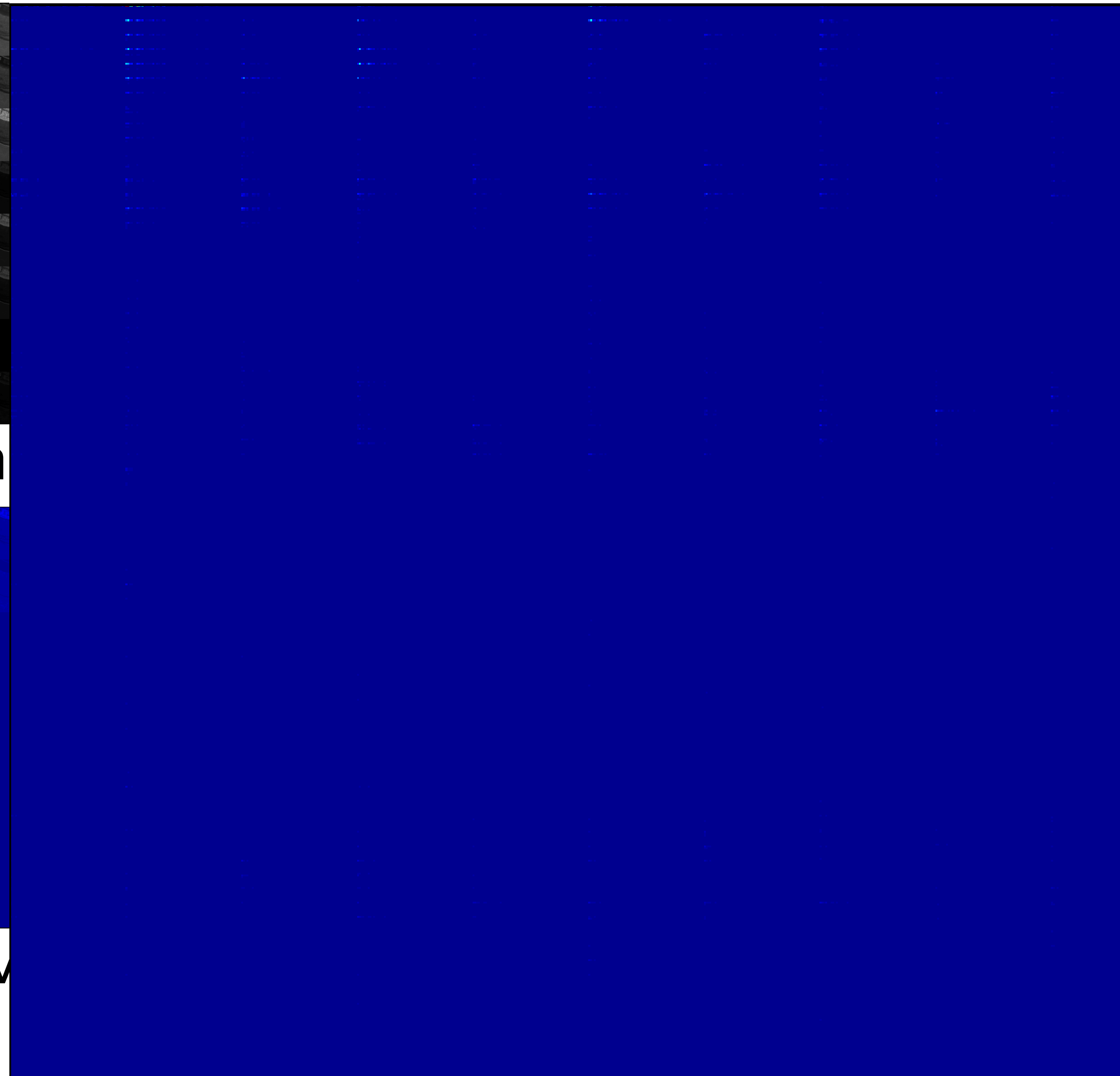
Example: Hyperspectral Data (AVIRIS)



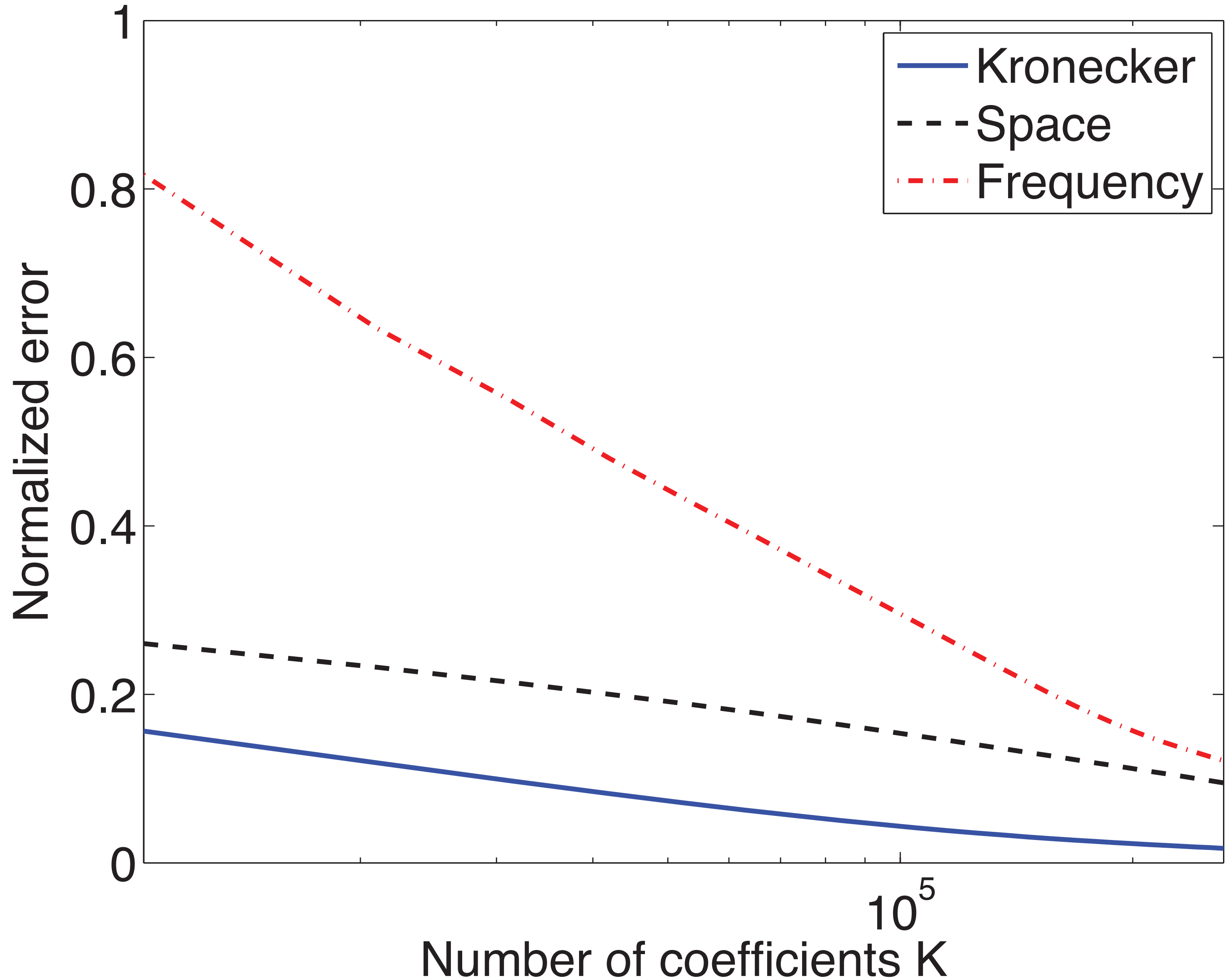
Original da



Spectral Wav

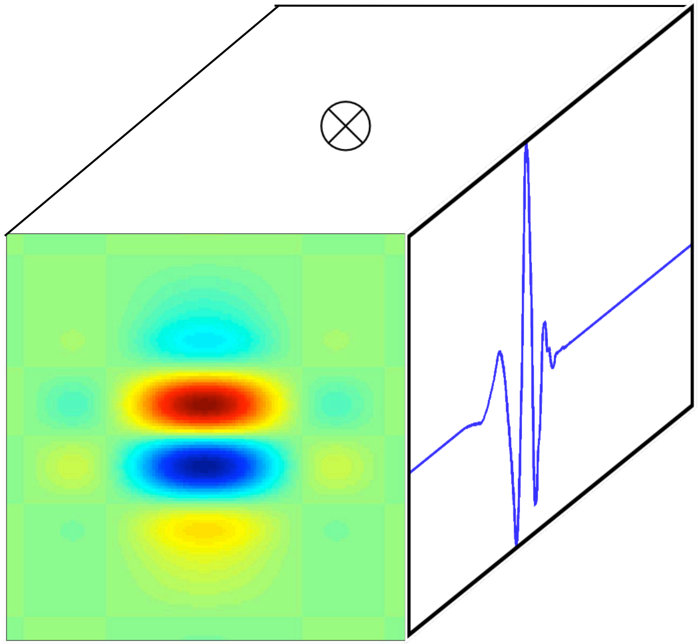


Example: Hyperspectral Data (AVIRIS)



$$\mathbf{x} \in \mathbb{R}^{128 \times 128 \times 128}$$

Kronecker CS with High-Dimensional Wavelets



- ***Theorem:*** KCS performs better than independent recovery along dimension e if

$$M < C\mu(\Phi_e, \Psi_e)^{\frac{\beta_e}{D-1}\left(1-\frac{\beta}{s_e}\right)} \prod_{d \neq e} \mu(\Phi_d, \Psi_d)$$

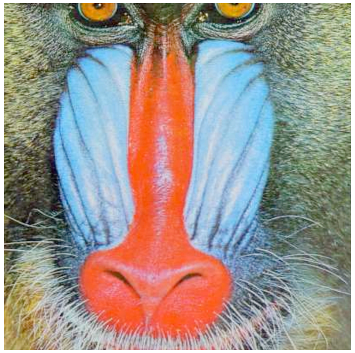
where

$$\beta = \frac{D}{2\sum_{d=1}^{D} 1/s_d} - \frac{1}{4},$$

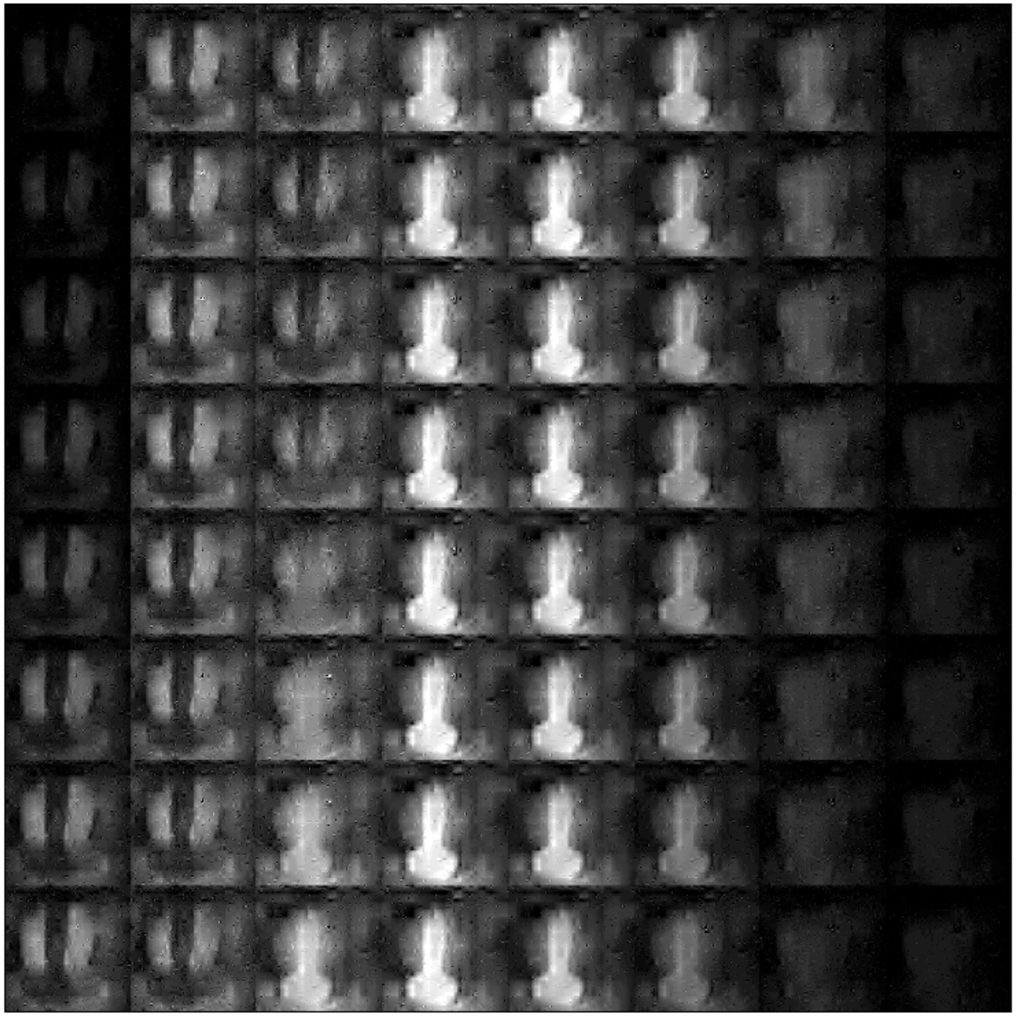
$$\beta_e = \frac{D-1}{2\sum_{d \neq e} 1/s_d} - \frac{1}{4}$$

In other words, KCS is ***most beneficial*** compared to independent CS when applied to dimension e with lowest smoothness s_e

Compressive Hyperspectral Imaging via Single Pixel Camera



[Measured by T. Sun, D. Takhar, K. Kelly]

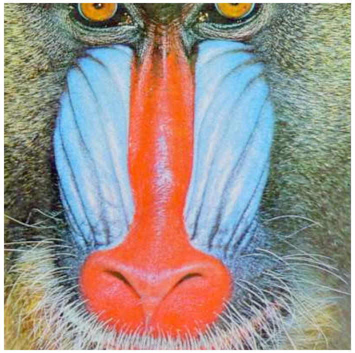


450nm-850nm range

$\mathbf{x} \in \mathbb{R}^{128 \times 128 \times 64}$

$M = 4096$ measurements per band

Compressive Hyperspectral Imaging via Single Pixel Camera



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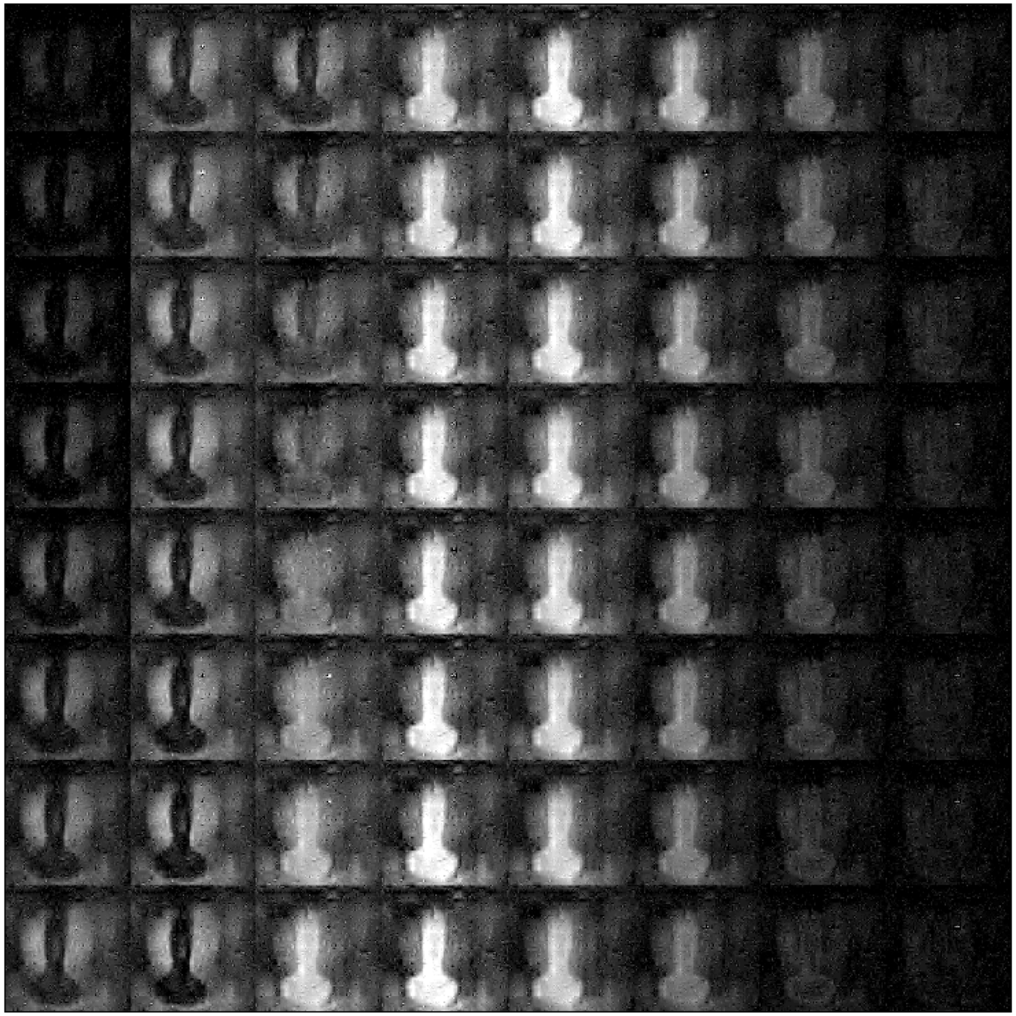
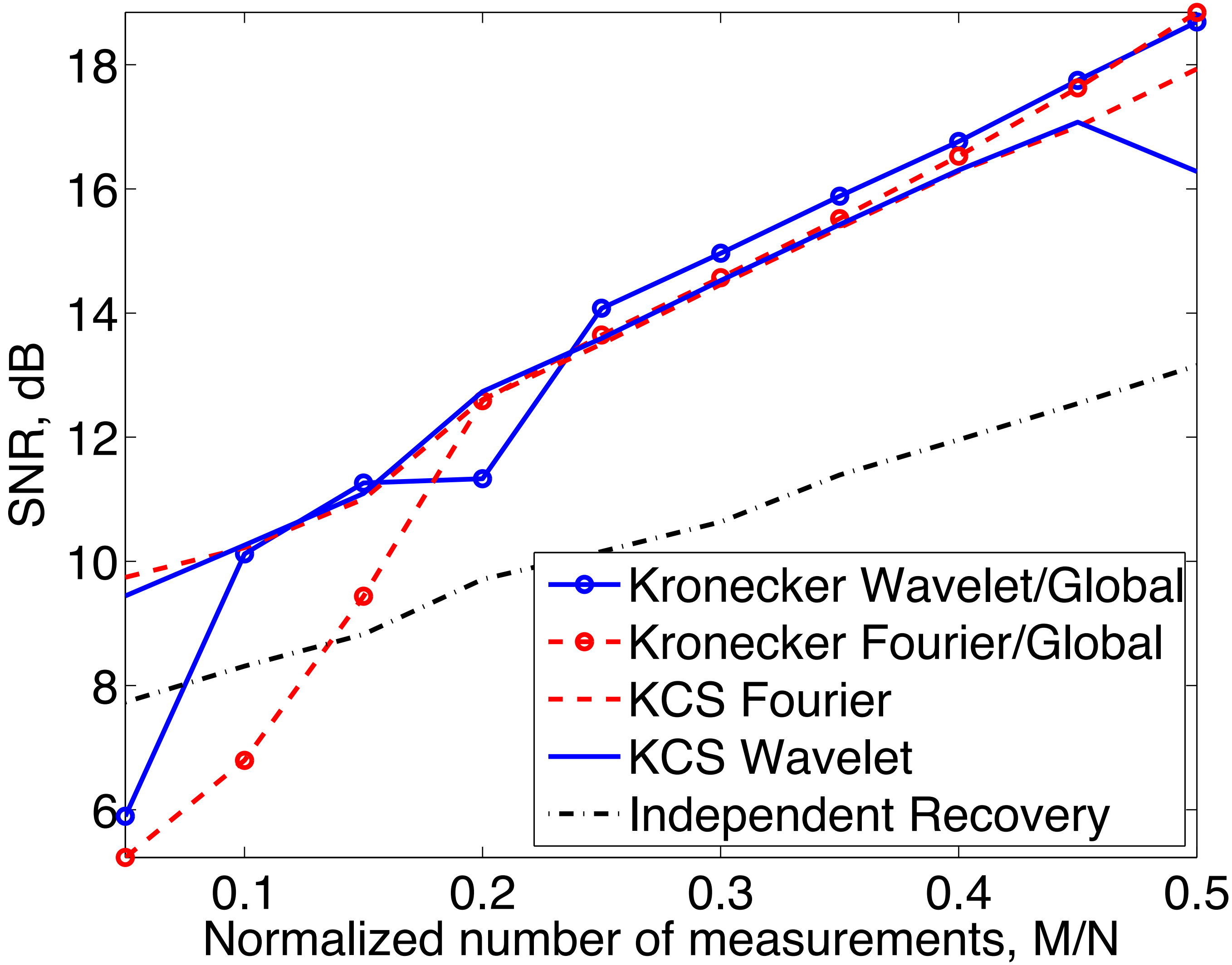


450nm-850nm range

$$\mathbf{x} \in \mathbb{R}^{128 \times 128 \times 64}$$

$M = 4096$ measurements per band - ***joint recovery***

Experimental Results: Hyperspectral Data (AVIRIS)



$$\mathbf{x} \in \mathbb{R}^{128\times128\times16}$$

Kronecker Compressive Sensing

- ***Succint*** mathematical framework for:
  - distributed measurement techniques
  - multidimensional signal compressibility
- Can be used with ***standard recovery algorithms***
- ***Same matrix metrics*** for CS suitability
- ***Same instance-optimality guarantees*** for sparse and compressible signals
- ***Mutually incoherent*** sparsity/measurement bases preferred for each dimension
- Certain bases ***transfer compressibility*** to higher dimensional ensembles through Kronecker products
- Future work
  - analysis of *suitable sparsity bases* for Kronecker products
  - *extensions* to additional applications (sparsity-based localization) [Cevher, Duarte, Baraniuk]