

Modelling Synchronization in TCP Networks Using Weakly-Coupled Oscillators

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Abstract—In this paper, we consider a fluid model of a TCP network consisting of many edge routers directing traffic through a common core link equipped with a small buffer. We are interested in how packet loss at this core router can dynamically couple, and potentially synchronize, the edge traffic. By using concepts from weakly-coupled oscillator theory, we show that synchronization occurs when a so-called coupling strength exceeds a critical value. This coupling strength is a property of the network, and can be expressed in terms of network parameters such as the core link’s utilization and its router’s buffer size. Avoiding synchronization requires a tradeoff between these two. That is, to avoid synchronization, small buffers should be accompanied by a commensurate reduction in link utilization. For example, in the paper, we will estimate that a 20-packet core buffer should have its link under-utilized by more than 35% to avoid synchronization.

I. INTRODUCTION

Some of the recent research in the congestion control of TCP networks has focused on the local stability of their fluid models [1] – [12]. The practical impact of this work has been to provide guidelines for designing protocols to adjust data-sending rates, and to actively manage packet loss, delay and throughput at congested links. As part of the research investigating the feasibility of using small-buffered core routers (on the order of tens of packets), see [13] – [17], researchers have studied the oscillatory behavior that emerges when network settings result in the fluid models becoming locally unstable; e.g., see [18]. Our research takes this approach one step further by presuming that the traffic in a fluid model is predominantly oscillatory and asks whether feedback signals from a congested core link can synchronize these flows. This research adds to prior research in network synchronization; e.g., [19] – [21], and introduces the concepts of *weakly-coupled oscillators* and *oscillator synchronization theory* [22] and [23]. Traditionally, synchronization in TCP networks has been viewed at the packet level, and concerned with the phase relationship between the sawtooth waveforms of individual congestion windows (resulting from TCP’s additive-increase, multiplicative decrease mechanism). That’s not the case here since we use fluid-model approximations to TCP [25]. Rather, we will assume that the fluid approximation of the congestion-window sizes are oscillatory, reflecting the local instability in the fluid-model’s differential equations, and

study whether these oscillations phase-synchronize because of feedback arising from a congested core link through which the fluid traffic passes. Our research aims to model and analyze this situation using synchronization theory which studies how coupling between intrinsic oscillators of disparate frequencies can ultimately synchronize. By analogy, the oscillating fluid flows act as the intrinsic oscillations, and packet loss at the common congested core link acts as the coupling mechanism.

In Section II of this paper we will introduce the network topology used in the rest of the paper. We refer to it as an edge-core topology and it consists of many sources, each passing through a distinct edge router, whose flows then aggregate to pass through a common core link. Each flow receives two congestion control signals. One from the packet loss incurred in passing through its own, and distinct, edge router, and a second congestion control signal from the core router through which all flows are assumed to traverse. In Section III we will introduce a phase model of the presumed oscillating fluid flows that will enable application of existing results on oscillator synchronization. To accomplish this, we will assume two things: (i) that the congestion signal received by a source is dominated by the packet loss at its own edge link, and (ii) in the absence of core congestion, each flow oscillates due to the local congestion feedback signal received from its congested edge router. Under these assumptions, the fluid-model equations will be interpreted as a set of weakly-coupled oscillators. These oscillators model the interaction between a source and its edge router. The “weak coupling” is a model of the weak congestion signal that each flow receives from the congested core link. In the fluid models, we will follow [14]-[17] and assume that both the edge and core routers have small buffers (on the order of tens of packets) and model their loss as that of an M/M/1 queue. In Theorem 1 we will state the main technical contribution of the paper, where the fluid model equations are distilled down to a set of coupled phase equations. “Phase” refers to the phase of an oscillating fluid flow’s periodic motion. In this theorem we will identify a key parameter in oscillator synchronization theory called the network’s *coupling strength*, and express it in terms of network parameters such as router buffer-size and link utilization. In Section IV we study the synchronization of the coupled phase equations. Synchronization means that the phase velocity of

all flows are equal, and that their pairwise phase-difference is stationary. We compute a so-called *critical coupling strength* for two realizations of the edge-core topology. Networks with coupling strength exceeding this critical value will be synchronized. Since the network's coupling strength and its critical value can be expressed in terms of network parameters, then an explicit formula, describing the onset of synchronization, can be equally expressed in terms of these parameters. In particular, this formula will describe a tradeoff between the core router's buffer size and its utilization when considering synchronization. In Section V we confirm the phase-model analysis via simulation of the original fluid model.

II. FLUID MODEL OF CONGESTION CONTROL IN AN EDGE-CORE TOPOLOGY

In Figure 1 we illustrate the so-called *edge-core topology* where N TCP flows pass through N distinct edge routers all passing through the same congested core link. Using [25], a

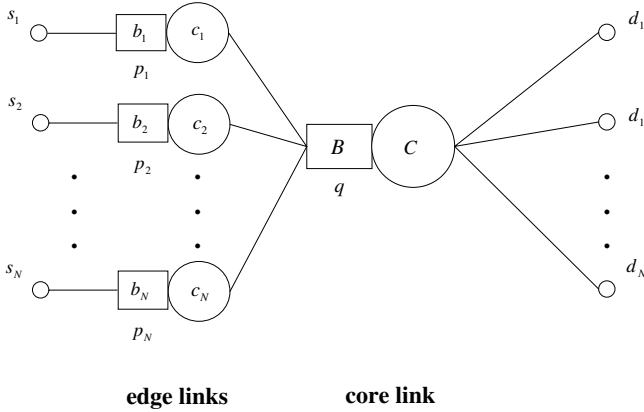


Fig. 1. An edge-core topology where N source-destination pairs (s_k, d_k) each have their traffic passing through an edge router with buffer-size b_k , link capacity c_k and loss p_k . All edge traffic passes through a congested core link with buffer-size B , link capacity C and loss q .

fluid model for the TCP-controlled source windows is

$$\dot{w}_k(t) = \frac{1}{\tau_k} - \frac{w_k(t)w_k(t - \tau_k)}{2\tau_k} [p_k(t - \tau_k) + q(t)] \quad (1)$$

where $k \in 1, 2, \dots, N$, and w_k is the k th congestion window experiencing a τ_k seconds of round-trip time. As in [14] we assume packet loss at both the edge and core routers to be that of an M/M/1 loss model so that:

$$p_k(t) = \left(\frac{\eta_k w_k(t)}{\tau_k C_k} \right)^{b_k}, \quad k = 1, \dots, N \quad (2)$$

and

$$q(t) = \left(\frac{\sum_{j=1}^N \frac{\eta_j w_j(t - \tau_j)}{\tau_j}}{C} \right)^B \quad (3)$$

where η_k is the number of concurrent sessions in the the k th flow, (b_k, c_k, p_k) describe the buffer size, link capacity and loss for the k th edge router and (B, C, q) the comparable parameters for the core link.

III. PHASE MODEL OF OSCILLATING TRAFFIC

In this section we first assume that TCP-controlled traffic levels oscillate, due solely to loss incurred at the edge links. This will allow us to consider (1)–(3) as a system of oscillators that are weakly coupled by the common loss signal received from a weakly-congested core link. The main technical contribution of the paper follows wherein the differential equations (1) – (3), describing coupled limit-cycle oscillators in the window states w_k , are distilled down to a set of coupled phase equations, where the phase $\theta_k \in [0, 2\pi)$ is the phase of w_k 's motion on the limit-cycle. The phase equations will then enable a study of oscillator synchronization. Key to this study is the coupling strength between oscillators, and in the development we will quantify the coupling strength in terms of buffer sizes and traffic intensities.

A. Weakly coupled oscillators

The fluid model equations in (1) can be written as

$$\dot{w}_k = f_{ek}(w_k) + \epsilon g_{ck}(w_k, w_{-k}) \quad (4)$$

where f_{ek} describes the window-control associated with the feedback from the k th edge router, g_{ck} the corresponding control mechanism associated with feedback from the core router, and the weak coupling term

$$\epsilon = \left(\frac{\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l}}{C} \right)^B.$$

For $\epsilon = 0$, the intrinsic behavior of the sources is

$$\dot{w}_k = f_{ek}(w_k)$$

which has equilibrium

$$w_k^* = \left(2 \left(\frac{c\tau}{\eta} \right)^{b_k} \right)^{\frac{1}{b_k+2}}. \quad (5)$$

In [18], a super-critical Hopf bifurcation was shown to exist for these intrinsic dynamics. The emergent, exponentially-stable limit-cycle has frequency and phase lag described by

$$\omega_k = \frac{\arccos\left(\frac{-1}{b_k+1}\right)}{\tau}; \quad \mu_k = \frac{1 - \sqrt{b_k(b_k+2)}}{w_k^* \arccos\left(\frac{-1}{b_k+1}\right) + 1}. \quad (6)$$

B. Phase motion of coupled oscillators

Bringing back the weak coupling term ϵg , we can now interpret (4) as a collection of weakly coupled oscillators. We wish to determine the threshold of coupling ϵ for which these oscillators experience synchronization. To do this we first determine the phase motion of these coupled limit-cycle oscillators. While a general phase model can be developed, we will focus on the case where the oscillators have almost equal coupling by assuming that the $\tau_k = \tau$, $b_k = b$, $\eta_k = \eta$ and $c_k = c$ up to $O(\epsilon)$. Then, $w_k^* = w^*$, $\omega_k = \omega$ and $\mu_k = \mu$ up to $O(\epsilon)$. In this case we have the following.

Theorem 1: (see appendix A) Assume the case of almost-equal coupling as described above. Then, the weakly-coupled limit-cycle oscillators in (4) have phase model:

$$\dot{\theta}_k = \omega_k - Kd + \frac{K}{N} \sum_{j=1}^N \sin(\theta_k - \theta_j + \alpha) + O(K^2) \quad (7)$$

where θ_k denotes the phase of the congestion-control window w_k ,

$$K = \frac{1}{|\cos(\pi_0)|} \frac{Bw^*}{2\tau} \left(\frac{\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l}}{C} \right)^B, \quad (8)$$

is the coupling strength and where

$$d = \frac{-\tan(\pi_0)}{B(b+2)} + \frac{1}{|\cos(\pi_0)|} \frac{b^2 + b - 1}{B(b+2)} \sin(\omega\tau + \pi_0);$$

and

$$\alpha = \omega\tau - \pi_0; \quad \pi_0 = \arctan(-\mu). \quad (9)$$

□

The remainder of the paper will focus on (7), and given that the source traffic intrinsically oscillates with frequencies ω_k , we will be interested in the threshold on coupling strength K , beyond which oscillations synchronize; i.e., $\dot{\theta}_k - \dot{\theta}_j = 0$. We will do this in the next section, and now show how K depends on the core and edge buffer sizes. We are particularly interested in conditions that result in small coupling strength.

C. Coupling strength as a function of buffer size

From (8), the coupling strength K can be expressed as a function of core buffer size B as in

$$K = \nu B \rho^B \quad (10)$$

where $\rho = \sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l C}$ is the core link's utilization and

$$\nu = \frac{w^*}{2\tau |\cos \pi_0|} \quad (11)$$

Contours of constant coupling strength are plotted in the (B, ρ) plane as shown in Figure 2. These plots show that for coupling to be weak, a small-buffered core must be underutilized. Conversely, we can express K as a function of edge parameters

$$K = \frac{B}{2\tau} \left(\frac{Nc}{C} \right)^B \frac{w^*}{|\cos \pi_0|} \left(\frac{\eta w^*}{c\tau} \right)^B \quad (12)$$

where the equilibrium window w^* , given in (5), is an explicit function of edge buffer size b . Again, we plot contours of constant K , but this time against b and the link capacity per flow $c\tau/\eta$. This is shown in Figure 3 where the coupling strength grows with edge buffer size b . This is consistent with Figure 2 since larger edge buffers increase the core links utilization, which from Figure 2 implies larger coupling strength.

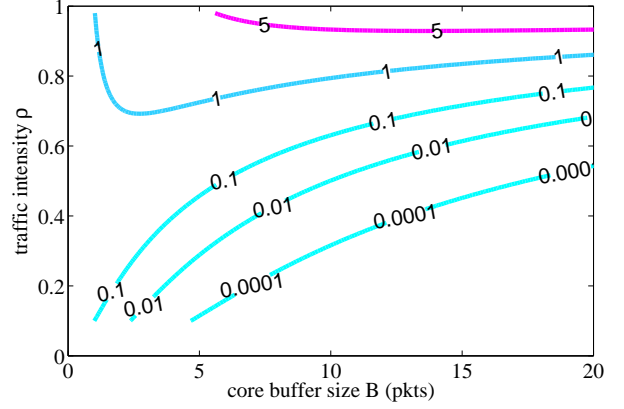


Fig. 2. The effect of core buffer size B and core utilization ρ on coupling strength K . The coupling strength is annotated on the constant- K contours.

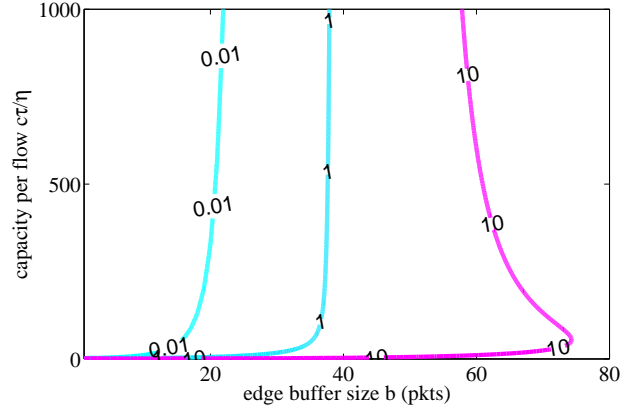


Fig. 3. The effect of edge buffer size b and link capacity per flow $c\tau/\eta$ on coupling strength K . The core link's buffer size is $B = 20$ pkts. The coupling strength is annotated on the constant- K contours.

IV. SYNCHRONIZATION

The phase model in (7) now allows us to address the question: *For what coupling strength K will the oscillating flows synchronize?* Synchronization means that oscillating flows achieve the same frequency $\lim_{t \rightarrow \infty} (\dot{\theta}_k - \dot{\theta}_j) = 0$, and necessarily, achieve stationary phase-difference: $\lim_{t \rightarrow \infty} (\theta_k - \theta_j) = \text{constant} \in [0, 2\pi]$. We will consider two cases of (7); when there are two oscillating flows ($N = 2$), and when there are an infinite number ($N \rightarrow \infty$). In each case, we will establish a *critical coupling strength* K_c , above which, synchronization is achieved. Together with (10) and (12), this will allow us to study synchronization as a function of parameters; e.g., core buffer-size and utilization. The case of an infinite number of oscillating flows falls into the well-studied class of coupled oscillators referred to as the Kuramoto model [22]. The particular variation on the Kuramoto model used here, see (7), is distinguished by the presence of phase delay α , and is most closely related to the work in [24].

A. Critical coupling

Consider the change of variable $\phi_k(t) = \theta_k(t) - Kdt$, so that (7) becomes

$$\dot{\phi}_k = \omega_k + \frac{K}{N} \sum_{j=1}^N \sin(\phi_k - \phi_j + \alpha). \quad (13)$$

We now consider the case when $N = 2$.

Proposition 1: (see appendix B for proof.) *Consider the phase model (13) for $N = 2$. Then, the critical coupling strength is*

$$K_c = \frac{\omega_2 - \omega_1}{\cos \alpha} \quad (14)$$

and the corresponding critical frequency is

$$\Omega_c = \frac{\omega_1 + \omega_2}{2} + \frac{K_c}{2} \sin \alpha \quad (15)$$

where ω_1 and ω_2 are given in (6) and α in (9). \square

Now we consider the case when $N \rightarrow \infty$ and, as in the Kuramoto model, assume that the intrinsic frequencies are drawn from a distribution $g(\omega)$; see [22] for details. A measure of synchronization is the *complex order parameter*

$$r(t)e^{j\psi(t)} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N e^{j\phi_k(t)},$$

where r is referred to as the *coherence*. If the oscillators remain unsynchronized, then at any given time their phases will be randomly distributed about the unit circle producing zero coherence. However, once the coupling strength exceeds a critical value, r becomes positive indicating that groups of oscillators are synchronizing. The hallmark of the Kuramoto model is that a further increase in coupling strength leads to rapid synchronization of the oscillator family.

Proposition 2: (see appendix C for proof) *Consider the phase model (13) when $N \rightarrow \infty$ and assume that the intrinsic frequencies w_k are selected from a given distribution $g(\omega)$. Then, the critical coupling strength is*

$$K_c = \frac{2|\cos \alpha|}{\pi g(\Omega_c)}, \quad (16)$$

where the critical frequency Ω_c satisfies

$$\int_0^\infty \frac{g(\Omega_c - x) - g(\Omega_c + x)}{2x} dx = \frac{\pi}{2} g(\Omega_c) \tan \alpha \quad \square$$

B. Tradeoff between buffer-size and link utilization

Combining the results of Theorem 1 and Proposition 2 allows us to derive an inequality describing those core buffer-sizes and link utilizations for which synchronization occurs. Synchronization of (13) occurs when $K > K_c$, which together with (10) and (16) gives a condition for synchronization:

$$\rho \geq \left(\frac{2|\cos \alpha|}{\pi g(\Omega_c) \nu B} \right)^{1/B} \quad (17)$$

where α and ν are defined in (9) and (11) respectively. To illustrate (17), we take $b = 20$ pkts, $c = 100$ pkts/s and $\tau = 0.155$ seconds. In Figure 4, we plot the traffic intensity ρ versus core buffer size B . The weakly-coupled phase model in (13) avoids synchronization provided their (B, ρ) pairs fall beneath the curve in Figure 4. In the next section we will consider simulations of the associated fluid model (1) – (3) to verify the accuracy of these phase model approximations.

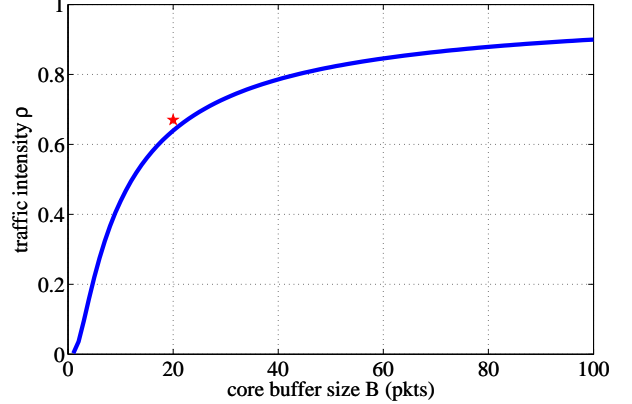


Fig. 4. Pairs (B, ρ) falling above this curve produce synchronization in a edge-core topology where $b = 20$ pkts, $c = 100$ pkts/s and $\tau = 0.155$ seconds. The \star corresponds to the network conditions for which the 60 edge router simulation in Section V.B is at the onset of synchronization.

V. SIMULATIONS

A. 2 Edge Routers

For calibration we simulate (1) – (3) for two edge routers ($N = 2$). The buffer sizes b and link capacity c of the edge routers are 20pkts and 1000pkts/s respectively. The buffer size for the core router is $B = 20$ pkts, the number of concurrent flows η is 10, $\tau_1 = 0.161$ s and $\tau_2 = 0.160$ s. Then, from (6), $\omega = 9.46$ rps and $\mu = -0.9176$ so that from (14), the critical coupling strength is $K_c = 0.0981$ which, using (8), occurs at a core link capacity of $C = 2190$ pkts/s. The simulation is conducted by decreasing the core link capacity from 2400pkts/s to 2000pkts/s, which from (8) increases the coupling. We observe that the source flows synchronize for a core link capacity of 2220pkts/s confirming Proposition 1.

B. 60 Edge Routers

In these simulations we consider a simulation of sixty edge routers, testing the tradeoff prediction made in Figure 4. The round-trip times τ_k are uniformly distributed in $[0.15, 0.155]$ s which effectively defines the distribution of intrinsic frequencies $g(\omega)$. The link capacity and buffer size of the sixty edge routers are fixed at 100pkts/s and 20pkts respectively. The buffer size of the core router is 20pkts. In the fluid model simulation of (1) – (3) we increase the core's link capacity from 6000pkts/s to 7900pkts/s and plot link utilization and loss for the edge and core routers in Figure 5 and Figure 6 respectively. In Figure 5, the dashed curves

denote the average utilization while the solid vertical lines denote the utilization's amplitude of oscillation. Similarly, in Figure 6 we show packet loss at the edge and core routers.

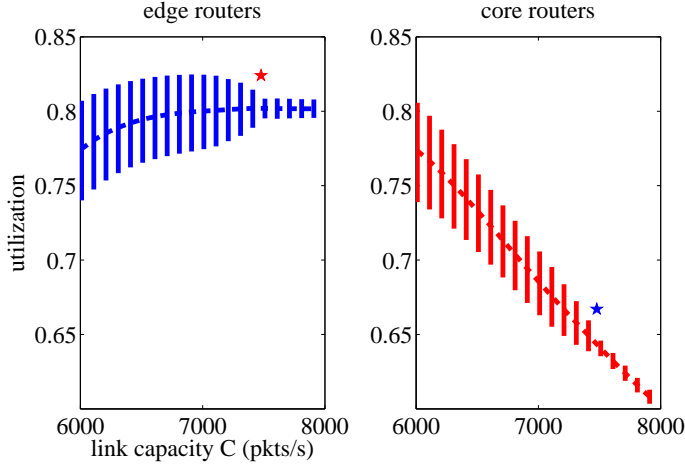


Fig. 5. Utilization versus core link capacity. The left plot shows edge link utilization and the right figure the core's utilization. Vertical bars denotes oscillation amplitude and synchronization. The \star denotes the onset of synchronization.

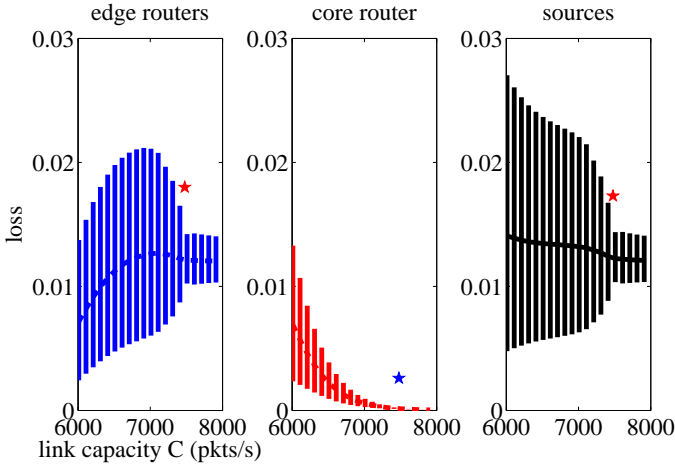


Fig. 6. Loss versus core link capacity. The left plot shows edge loss, the middle plot shows core loss and the right figure the loss experienced at the sources. The \star denotes the onset of synchronization.

To verify the tradeoff prediction made in Figure 4, we observe from Figure 5 and Figure 6 that the onset of synchronization occurs at approximately 7478pkts/s; see the stars in these plots. A corresponding \star in the rightmost plot of Figure 5 marks the 64% utilization occurring at this 7478pkts/s core link capacity. Finally, returning to Figure 4 we see that the onset of synchronization is predicted to occur at a 64% utilization when $B = 20$ pkts/s. Thus in this simulation, Figure 4 accurately models the onset of synchronization in (B, ρ) space.

VI. CONCLUSION

In this paper we analyzed a fluid model for congestion control in a so-called edge-core topology. The edge traffic was assumed to be oscillating and we studied its synchronization at the core congested link. We developed a phase model (7) for the congestion control dynamic which allowed us to frame the network synchronization problem as one involving weakly-coupled oscillators, and to express the coupling strength in terms of network parameters; particularly, the core link's utilization and buffer size. We considered two special cases of (7); the case of two edge routers ($N = 2$), and the case of many ($N \rightarrow \infty$). The latter case corresponds to a delayed-version of the Kuramoto model and in (17) showed that synchronization can be avoided if

$$\rho < \left(\frac{\kappa}{B}\right)^{1/B}$$

where ρ and B are the core link's traffic intensity and buffer size respectively, and where κ is a constant depending on the intrinsic edge traffic oscillations. This tradeoff between link utilization and buffer-size was confirmed on the original fluid model for $N = 60$ edge routers, and showed that for a core buffer size of $B = 20$ pkts, a link utilizations less than 65% was required to avoid synchronization. This is consistent with [15] where a 75% link utilization is proposed.

VII. APPENDIX

A. The development of phase model for weakly coupled TCP flows

Lemma 1: For N TCP flows in the edge-core topology shown in (1), if the coupling is weak, then the phase model is

$$\begin{aligned} \frac{d\theta_k}{dt} = & \omega_k - \left(\frac{\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l}}{C}\right)^B \frac{w_k^*}{2\tau_k} \cdot \\ & \left(\frac{\mu_k}{b_k + 2} + (b_k - 1 + \frac{1}{b_k + 2}) \sin(\omega_k \tau_k) - \right. \\ & \left. \mu_k (b_k - 1 + \frac{1}{b_k + 2}) \cos(\omega_k \tau_k)\right) - \\ & \left(\frac{\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l}}{C}\right)^B \frac{B(w_k^*)^2}{2\tau_k} \left(\frac{\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l}\right)^{-1} \sum_{j=1}^N \left(\frac{\eta_j \hat{r}_j}{\tau_j \hat{r}_k} \cdot \right. \\ & \left. (\sin(\theta_j - \theta_k - \omega_j \tau_j) - \mu_k \cos(\theta_j - \theta_k - \omega_j \tau_j))\right), \quad (18) \end{aligned}$$

where $\theta_k(t)$ denotes the phase of window $w_k(t)$, ω_k is the intrinsic frequency of flow k , w_k^* is the equilibrium window size k , η_k , τ_k are the concurrent flow number and round-trip time delay for k , respectively, c_k , b_k are the link capacity and buffer size for edge router k , C and B are the link capacity and buffer size for core router, \hat{r}_k is the amplitude of oscillation for flow k and

$$\mu_k = \frac{1 - \sqrt{b_k(b_k + 2)}}{w_k^* \arccos\left(\frac{-1}{b_k + 1}\right) + 1}.$$

Proof: For sake of convenience, we define

$$f_k = \frac{1}{\tau_k} - \frac{w_k(t)w_k(t-\tau_k)}{2\tau_k} \left(\frac{\eta_k w_k(t-\tau_k)}{\tau_k c_k} \right)^{b_k}$$

$$g_k = -\frac{w_k(t)w_k(t-\tau_k)}{2\tau_k} \left(\frac{\sum_{j=1}^N \frac{\eta_j w_j(t-\tau_j)}{\tau_j}}{\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l}} \right)^B$$

and

$$\epsilon = \left(\frac{\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l}}{C} \right)^B. \quad (19)$$

Such that (??) becomes

$$\frac{dw_k(t)}{dt} = f_k(w_k(t), w_k(t-\tau_k)) + \epsilon g_k(w_k(t), w_1(t-\tau_1), \dots, w_N(t-\tau_N)). \quad (20)$$

Since [18] shown that when $\epsilon = 0$, the uncoupled system has a sub-critical Hopf bifurcation at the emergence of oscillations. Such that the uncoupled system has an exponentially orbitally stable ω_k -periodic solution $\gamma_k \in \mathcal{R}$. So we use the phase of $\gamma_k(t)$, i.e., $\theta_k(t)$, to describe the limit cycle γ_k . From [23], a direct product of exponentially asymptotically stable limit cycle attractors, $M = \gamma_1 \times \dots \times \gamma_N$, is a normally hyperbolic compact invariant manifold for the uncoupled system (20). Since the invariant manifold persists for $\epsilon > 0$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$ system (20) has a normally hyperbolic invariant manifold in an ϵ -neighborhood of M . Let us denote

$$w_k(t) = \gamma_k(t + \varphi_k(\kappa)) + \epsilon P_k(t, \varphi, \epsilon) \quad (21)$$

where ϵP_k account for the ϵ -perturbation of the invariant manifold M , $\varphi_k = \frac{\theta_k}{\omega_k} - t$, which is called the phase deviation, and $\kappa = \epsilon t$. For the sake of convenience we denote $\varphi_k(\kappa)$ simply by φ_k . Then, equalize the derivative of (21) and (20), yield

$$\begin{aligned} \frac{dw_k(t)}{dt} &= \frac{d\gamma_k(t + \varphi_k)}{dt} \left(1 + \epsilon \frac{d\varphi_k}{d\kappa} \right) + \epsilon \frac{\partial P_k(t, \varphi, \epsilon)}{\partial t} \\ &= f_k(\gamma_k(t + \varphi_k), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k))) + \\ &\quad \epsilon g_k(\gamma_k(t + \varphi_k(\kappa)), \gamma(t - \tau + \varphi(\kappa - \epsilon\tau))) + \\ &\quad \epsilon D_x(f_k(\gamma_k(t + \varphi_k(\kappa)), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)))) \cdot \\ &\quad P_k(t, \varphi, \epsilon) + \epsilon P_k(t - \tau_k, \varphi(\kappa - \epsilon\tau), \epsilon) \cdot \\ &\quad D_y(f_k(\gamma_k(t + \varphi_k(\kappa)), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)))) \end{aligned}$$

plus terms of order $O(\epsilon^2)$, where D_x and D_y are defined as the partial derivatives for function $f(x, y)$, i.e., $D_x(f(x, y)) = \frac{\partial f_k(x, y)}{\partial x}$ and $D_y(f(x, y)) = \frac{\partial f_k(x, y)}{\partial y}$. Since

$$\frac{d\gamma_k(t + \varphi_k)}{dt} = f_k(\gamma_k(t + \varphi_k), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k))),$$

we obtain

$$\begin{aligned} &f_k(\gamma_k(t + \varphi_k), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k))) \frac{d\varphi_k}{d\kappa} + \\ &\frac{\partial P_k(t, \varphi, 0)}{\partial t} \\ &= g_k(\gamma_k(t + \varphi_k(\kappa)), \gamma(t - \tau + \varphi(\kappa - \epsilon\tau))) + \\ &\quad D_x(f_k(\gamma_k(t + \varphi_k(\kappa)), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)))) \cdot \\ &\quad P_k(t, \varphi, 0) + P_k(t - \tau_k, \varphi(\kappa - \epsilon\tau), 0) \cdot \\ &\quad D_y(f_k(\gamma_k(t + \varphi_k(\kappa)), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)))). \end{aligned}$$

Since the uncoupled system is at the emergence of oscillation, i.e.,

$$\gamma_k(t) = w_k^* + r_k(t),$$

where w_k^* is the equilibrium window size for flow k . Using Taylor expansion for f_k , g_k , $D_x(f_k)$ and $D_y(f_k)$, yield

$$\begin{aligned} &f_k(\gamma_k(t + \varphi_k), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k))) \\ &= f_k(w_k^*, w_k^*) + D_x(f_k(w_k^*, w_k^*))r_k(t + \varphi_k) + \\ &\quad D_y(f_k(w_k^*, w_k^*))r_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)) + O(r^2) \\ &= D_x(f_k(w_k^*, w_k^*))r_k(t + \varphi_k) + \\ &\quad D_y(f_k(w_k^*, w_k^*))r_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)) + O(r^2), \end{aligned}$$

since $f_k(w_k^*, w_k^*) = 0$.

$$\begin{aligned} &g_k(\gamma_k(t + \varphi_k(\kappa)), \gamma(t - \tau + \varphi(\kappa - \epsilon\tau))) \\ &= g_k(w_k^*, w^*) + D_x(g(w_k^*, w^*))r_k(t + \varphi_k) + \\ &\quad D_y(g(w_k^*, w^*))r(t - \tau + \varphi(\kappa - \epsilon\tau)) + O(r^2). \end{aligned}$$

$$\begin{aligned} &D_x(f_k(\gamma_k(t + \varphi_k(\kappa)), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)))) \\ &= D_x(f_k(w_k^*, w_k^*)) + D_x(D_x(f_k(w_k^*, w_k^*)))r_k(t + \varphi_k(\kappa)) + \\ &\quad D_y(D_x(f_k(w_k^*, w_k^*)))r_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)) + O(r^2), \end{aligned}$$

and

$$\begin{aligned} &D_y(f_k(\gamma_k(t + \varphi_k(\kappa)), \gamma_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)))) \\ &= D_y(f_k(w_k^*, w_k^*)) + D_x(D_y(f_k(w_k^*, w_k^*)))r_k(t + \varphi_k(\kappa)) + \\ &\quad D_y(D_y(f_k(w_k^*, w_k^*)))r_k(t - \tau_k + \varphi_k(\kappa - \epsilon\tau_k)) + O(r^2) \end{aligned}$$

If the coupling strength $\epsilon \ll \frac{1}{\tau}$ such that $\epsilon\tau \ll 1$, we have

$$\begin{aligned} &\frac{\partial P_k(t, \varphi, 0)}{\partial t} + \frac{d\varphi_k}{d\kappa} (D_x(f_k(w_k^*, w_k^*))r_k(t + \varphi_k) + \\ &\quad D_y(f_k(w_k^*, w_k^*))r_k(t - \tau_k + \varphi_k)) \\ &= (D_x(f_k(w_k^*, w_k^*)) + D_x(D_x(f_k(w_k^*, w_k^*)))r_k(t + \varphi_k(\kappa))) + \\ &\quad D_y(D_x(f_k(w_k^*, w_k^*)))r_k(t - \tau_k + \varphi_k) P_k(t, \varphi, 0) \\ &\quad + (D_y(f_k(w_k^*, w_k^*)) + D_x(D_y(f_k(w_k^*, w_k^*)))r_k(t + \varphi_k(\kappa))) + \\ &\quad D_y(D_y(f_k(w_k^*, w_k^*)))r_k(t - \tau_k + \varphi_k) P_k(t - \tau_k, \varphi, 0) \\ &\quad + g_k(w_k^*, w^*) + D_x(g(w_k^*, w^*))r_k(t + \varphi_k) + \\ &\quad D_y(g(w_k^*, w^*))r(t - \tau + \varphi). \end{aligned}$$

So we get

$$\frac{dP_k(t, \varphi, 0)}{dt} = A_{k1}P_k(t, \varphi, 0) + A_{k2}P_k(t - \tau_k, \varphi, 0) + \bar{e}_k(t, \varphi), \quad (22)$$

Vector φ are treated as parameters here. As for TCP/Droptail in (??),

$$A_{k1} = - \left(\frac{1}{w_k^* \tau_k} + \frac{b_k + 1}{\tau_k (w_k^*)^2} r_k(t - \tau_k + \varphi_k) \right),$$

$$A_{k2} = - \frac{b_k + 1}{w_k^* \tau_k} - \frac{b_k + 1}{\tau_k (w_k^*)^2} r_k(t + \varphi_k) - \frac{b_k(b_k + 1)}{\tau_k (w_k^*)^2} r_k(t - \tau_k + \varphi_k),$$

and

$$\begin{aligned} & \bar{e}_k(t, \varphi) \\ &= \frac{-1}{2\tau_k} \left((w_k^*)^2 + w_k^* r_k(t + \varphi_k) \right) + w_k^* r_k(t - \tau_k + \varphi_k) + \\ & B(w_k^*)^2 \left(\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l} \right)^{-1} \sum_{j=1}^N \frac{\eta_j}{\tau_j} r_j(t - \tau_j + \varphi_j) \Bigg) + \\ & \frac{1}{w_k^* \tau_k} (r_k(t + \varphi_k) + (b_k + 1)r_k(t - \tau_k + \varphi_k)) \frac{d\varphi_k}{d\kappa}. \end{aligned}$$

Define $P_k(t, \varphi, 0) = -\frac{(w_k^*)^3}{2(b_k+2)} + y_k(t, \varphi)$, (22) becomes

$$\frac{dy_k(t, \varphi)}{dt} = A_{k1} y_k(t, \varphi) + A_{k2} y_k(t - \tau_k, \varphi) + e_k(t, \varphi), \quad (23)$$

where $y_k(t, \varphi)$ is an unknown vector variable, and

$$\begin{aligned} & e_k(t, \varphi) \\ &= \frac{-w_k^*}{2\tau_k} \left(\frac{1}{b_k + 2} r_k(t + \varphi_k) - \right. \\ & \left. (b_k - 1 + \frac{1}{b_k + 2}) r_k(t - \tau_k + \varphi_k) + \right. \\ & \left. B w_k^* \left(\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l} \right)^{-1} \sum_{j=1}^N \frac{\eta_j}{\tau_j} r_j(t - \tau_j + \varphi_j) \right) \\ & + \frac{1}{w_k^* \tau_k} (r_k(t + \varphi_k) + (b_k + 1)r_k(t - \tau_k + \varphi_k)) \frac{d\varphi_k}{d\kappa}. \end{aligned}$$

Since the uncoupled systems are at the emergence of oscillations,

$$-\frac{1}{w_k^* \tau_k} (r_k(t + \varphi_k) + (b_k + 1)r_k(t - \tau_k + \varphi_k)) = \dot{r}_k(t + \varphi_k).$$

Suppose the fundamental component for $r_j(t)$ is $r_j(t) = \hat{r}_j \sin(\omega_j t)$, where \hat{r}_j is the amplitude of the emergent oscillation for flow j , such that

$$\begin{aligned} & e_k(t, \varphi) \\ &= -\cos(\omega_k(t + \varphi_k)) \cdot \\ & \left(\hat{r}_k \omega_k \frac{d\varphi_k}{d\kappa} + \frac{\hat{r}_k w_k^*}{2\tau_k} (b_k - 1 + \frac{1}{b_k + 2}) \sin(\omega_k \tau_k) \right) - \\ & \cos(\omega_k(t + \varphi_k)) \frac{B(w_k^*)^2}{2\tau_k} \left(\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l} \right)^{-1} \cdot \\ & \sum_{j=1}^N \frac{\eta_j \hat{r}_j}{\tau_j} (\sin(\omega_j(t + \varphi_j)) - \omega_k(t + \varphi_k) - \omega_j \tau_j) - \end{aligned} \quad (24)$$

$$\begin{aligned} & \sin(\omega_k(t + \varphi_k)) \left(\frac{\hat{r}_k w_k^*}{2\tau_k (b_k + 2)} - \right. \\ & \left. \frac{\hat{r}_k w_k^*}{2\tau_k} (b_k - 1 + \frac{1}{b_k + 2}) \cos(\omega_k \tau_k) \right) - \\ & \sin(\omega_k(t + \varphi_k)) \frac{B(w_k^*)^2}{2\tau_k} \left(\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l} \right)^{-1} \cdot \\ & \sum_{j=1}^N \frac{\eta_j \hat{r}_j}{\tau_j} (\cos(\omega_j(t + \varphi_j)) - \omega_k(t + \varphi_k) - \omega_j \tau_j) \quad (25) \end{aligned}$$

Because e_k , A_{1k} and A_{2k} are ω_k -periodic, if (23) has a non-trivial ω_k -periodic solution, according to the theorem 4.22 in [27], we must have

$$\int_{t_0}^{t_0 + \frac{2\pi}{\omega_k}} x_k(t, \varphi_k) e_k(t, \varphi) dt = 0, \quad (26)$$

where x_k is the solution of the adjoint homogeneous system of (23), i.e

$$\dot{x}_k(t) = -A_{k1}(t)x_k(t) - A_{k2}(t + \tau_k)x_k(t + \tau_k). \quad (27)$$

Because $r_k = \hat{r}_k \sin(\omega_k t)$ and $\omega_k \tau_k = \arccos(\frac{-1}{b_k+1})$, thus

$$\begin{aligned} & A_{k1}(t) = \frac{-1}{w_k^* \tau_k} \cdot \\ & \left(1 - \frac{\hat{r}_k}{w_k^*} \sin(\omega_k t) - \frac{\sqrt{b_k(b_k + 2)}}{w_k^*} \hat{r}_k \cos(\omega_k t) \right) \quad (28) \end{aligned}$$

and

$$\begin{aligned} & A_{k2}(t + \tau_k) \\ &= \frac{-1}{w_k^* \tau_k} \left(b_k + 1 + \frac{\hat{r}_k}{w_k^*} (b_k^2 + b_k - 1) \sin(\omega_k t) + \right. \\ & \left. \frac{\sqrt{b_k(b_k + 1)}}{w_k^*} \hat{r}_k \cos(\omega_k t) \right). \quad (29) \end{aligned}$$

Suppose the fundamental component of $x_k(t)$ is

$$x_k(t) = g_0 + f_1 \sin(\omega_k t) + g_1 \cos(\omega_k t). \quad (30)$$

Then

$$\begin{aligned} & x_k(t + \tau_k) \\ &= g_0 - \left(\frac{f_1}{b_k + 1} + \frac{g_1 \sqrt{b_k(b_k + 2)}}{b_k + 1} \right) \cos(\omega_k t) - \\ & \left(\frac{g_1}{b_k + 1} - \frac{f_1 \sqrt{b_k(b_k + 2)}}{b_k + 1} \right) \sin(\omega_k t). \quad (31) \end{aligned}$$

Then, we substitute (28) – (31) into (27) and compare the coefficient of $\cos(\omega_k t)$, we obtain

$$f_1 \omega_k = \frac{1}{w_k^* \tau_k} \left(g_1 - (b_k + 1) \left(\frac{f_1}{b_k + 1} + \frac{g_1 \sqrt{b_k(b_k + 2)}}{b_k + 1} \right) \right),$$

which means that

$$\frac{f_1}{g_1} = \frac{1 - \sqrt{b_k(b_k + 2)}}{w_k^* \arccos(\frac{-1}{b_k+1}) + 1}.$$

So

$$\frac{\int_{t_0}^{t_0 + \frac{2\pi}{\omega_k}} x_k(t + \varphi_k) \sin(\omega_k(t + \varphi_k)) dt}{\int_{t_0}^{t_0 + \frac{2\pi}{\omega_k}} x_k(t + \varphi_k) \cos(\omega_k(t + \varphi_k)) dt} = \frac{f_1}{g_1} := \mu_k,$$

i.e.,

$$\mu_k = \frac{1 - \sqrt{b_k(b_k + 2)}}{w_k^* \arccos\left(\frac{-1}{b_k + 1}\right) + 1}. \quad (32)$$

To substitute (25) and (31) into the (26), produce

$$\begin{aligned} & \omega_k \frac{d\varphi_k}{d\kappa} \\ = & -\frac{w_k^*}{2\tau_k} \left(\frac{\mu_k}{b_k + 2} + (b_k - 1 + \frac{1}{b_k + 2}) \sin(\omega_k \tau_k) - \right. \\ & \left. \mu_k (b_k - 1 + \frac{1}{b_k + 2}) \cos(\omega_k \tau_k) \right) - \\ & \frac{B(w_k^*)^2}{2\tau_k} \left(\sum_{l=1}^N \frac{\eta_l w_l^*}{\tau_l} \right)^{-1} \cdot \\ & \sum_{j=1}^N \frac{\eta_j \hat{r}_j}{\tau_j \hat{r}_k} (\sin(\omega_j(t + \varphi_j) - \omega_k(t + \varphi_k) - \omega_j \tau_j) - \\ & \mu_k \cos(\omega_j(t + \varphi_j) - \omega_k(t + \varphi_k) - \omega_j \tau_j)). \end{aligned}$$

Since $\theta_j(t) = \omega_j(t + \varphi_j)$, we get the weakly coupled phase model for TCP flows.

B. Proof for Proposition 1

Proof: for $N = 2$, subtracting $\dot{\phi}_1$ by $\dot{\phi}_2$ yields

$$\dot{\phi}_1 - \dot{\phi}_2 = \omega_1 - \omega_2 + K \cos \alpha \sin(\phi_1 - \phi_2).$$

If the phases of ϕ_1 and ϕ_2 get locked, i.e., $\phi_1 - \phi_2 = \text{constant}$, which means these 2 TCP flows get synchronized, there must satisfy

$$K \geq \frac{\omega_2 - \omega_1}{\cos \alpha} := K_c,$$

such that

$$\phi_1 - \phi_2 = \arcsin\left(\frac{\omega_2 - \omega_1}{K \cos \alpha}\right).$$

So for $K = K_c$, $\phi_1 - \phi_2 = \frac{\pi}{2}$. Then from (13), we obtain (15).

C. Proof for Proposition 2

Proof: Let us assume $\alpha \in [\frac{-\pi}{2}, \frac{\pi}{2}]$. If not, suppose $\alpha \in [\frac{\pi}{2}, \pi]$, then (13) becomes

$$\frac{d\phi_k}{dt} = \omega_k - \frac{K}{N} \sum_{j=1}^N \sin(\phi_k(t) - \phi_j(t) + \alpha - \pi), \quad (33)$$

which is exactly the system studied in [28]. So, we only work on $\alpha \in [\frac{-\pi}{2}, \frac{\pi}{2}]$. It is not difficult to show that the analysis is also similar to [28]. So for all $\alpha \in [\frac{-\pi}{2}, \pi]$, we have (16).

REFERENCES

- [1] F. P. Kelly, A. K. Maulloo and D. Tan, "Rate Control for Communication Networks: Shadow Prices, Proportional Fairness and Stability," *Journal of the Operational Research Society*, Vol.49, pp. 237-252, 1998.
- [2] R. Johari and D. Tan, "End-to-End Congestion Control for the Internet: Delays and Stability," *IEEE/ACM Transactions on Networking* Vol. 9, no. 6, pp. 818-832, 2001.
- [3] S. Athuraliya, V.H. Li, S. Low and Q. Yin, "REM: Active Queue Management," *IEEE Network*, Vol.15, no.3, pp. 48-53, 2001.
- [4] S. Kunniyur and R. Srikant, "An adaptive virtual queue (AVQ) algorithm for active queue management," *IEEE/ACM Transactions on Networking (TON)*, Vol.12, no.2, pp. 286-299, 2004.
- [5] C.V. Hollot, V. Misra, D. Towsley and W. Gong, "Analysis and Design of Controllers for AQM Routers Supporting TCP Flows," *IEEE Transactions on Automatic Control*, Vol.47, no.6, pp. 945-959, 2002.
- [6] F. Paganini, J. Doyle and S. Low, "Scalable Laws for Stable Network Congestion Control," *Proceedings of the IEEE Conference on Decision and Control*, Vol.1, pp. 185-190, 2001.
- [7] G. Vinnicombe, "On the Stability of Networks Operating TCP-Like Congestion Control," *Proceedings of the IFAC World Congress*, Barcelona, 2002.
- [8] G. Vinnicombe, "Robust Congestion Control for Internet," available at <http://www-control.eng.cam.ac.uk/gv/internet/index.html>, 2002.
- [9] S. Kunniyur and R. Srikant, "Stable, Scalable, Fair Congestion Control and AQM Schemes that Achieve High Utilization in the Internet," *IEEE Transactions on Automatic Control*, Vol. 48, no. 11, pp. 2024- 2028, 2003.
- [10] R. Srikant, *The Mathematics of Internet Congestion Control*, System & Control: Foundations and Applications Series, T. Basar, Ed., Birkhäuser, 2003.
- [11] H. Han, C.V. Hollot, Y. Chait and V. Misra, "TCP Networks Stabilized by Buffer-based AQMs," *INFOCOM*, 2004.
- [12] P. Ranjan, R. J. La, and E. H. Abed, "Characterization of global stability conditions for rate control with an arbitrary communication delay," *IEEE/ACM Trans. on Networking*, Vol.14, no.1, pp. 94-107, 2006.
- [13] G. Appenzeller, I. Keslassy, N. McKeown, "Sizing Router Buffers," *SIGCOMM*, 2004.
- [14] G. Raina and D. Wischik, "Buffer Sizes for Large Multiplexers: TCP Queueing Theory and Instability Analysis", *EuroNGI Conference on Next Generation Internet Networks*, pp. 173- 180, 2005.
- [15] D. Wischik, "Buffer requirements for high-speed routers," *ECOC*, 2005.
- [16] D. Wischik and N. McKeown, "Part I: Buffer sizes for core routers," *em ACM/SIGCOMM CCR*, 2005.
- [17] G. Raina, D. Towsley and D. Wischik, "Part II: Control theory for buffer sizing," *ACM/SIGCOMM CCR*, 2005.
- [18] G. Raina, "Local Bifurcation Analysis of Some Dual Congestion Control Algorithms," *IEEE Transactions of Automatic Control*, Vol. 50, no. 8 pp. 1135- 1146, 2005.
- [19] F. Baccelli and D. Hong, "Interaction of TCP flows as billiards," *INFOCOM 2003*, Vol.2, no. 3, pp. 895- 905, 2003.
- [20] J.P. Hespanha, "Stochastic Hybrid Systems: Application to Communication Networks," *Lecture Notes in Computer Science*, Springer Verlag, Vol. 2993, pp. 387 - 401, 2004.
- [21] R. Shorten, F. Wirth, and D. Leith, "A positive systems model of tcp-like congestion control: Asymptotic results," *Hamilton Institute, Tech. Rep.*, 2004-1, April 2004.
- [22] S.H. Strogatz, "From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators," *Physica D: Nonlinear Phenomena*, Vol. 143, no.1, pp. 1-20, 2000.
- [23] F. C. Hoppensteadt, E. M. Izhikevich, *Weakly Connected Neural Networks*, Applied Mathematical Sciences 126, Springer, 1997.
- [24] Eugene M. Izhikevich, "Phase models with explicit time delays," *Physical Review E*, Vol. 58, no. 1, pp. 905-908, 1998.
- [25] V. Misra, W. B. Gong, and D. Towsley, "Fluid-based Analysis of a Network of AQM Routers Supporting TCP Flows with an Application to RED," *Proceedings of ACM/SIGCOMM*, 2000.
- [26] H.Han, C.V. Hollot and D. Towsley, "Modelling Synchronization in TCP Networks Using Weakly-Coupled Oscillators," <http://www.ecs.umass.edu/ece/hollot/CongestionControl/INFOCOM07TechReport.pdf>
- [27] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, 1966.

- [28] H. Sakaguchi, Y. Kuramoto, "A Solvable Active Rotator Model Showing Phase Transitions via Mutual Entrainment," *Progress of Theoretical Physics*, Vol.76, no.3, 1986.