# Fundamental Limits in Detecting Whether a Signal Has Been Quantized

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***Abstract*—It is often important to know whether a received signal was sent directly by a friend, or was recorded by an adversary and then replayed. In this paper, we use a characteristic of the recording, namely the quantization, to study if the replayed (quantized) signal can be detected. In particular, we consider the requirements on the quantizer to keep the quantization from being detected in additive Gaussian noise. If a signal with length $m$ is sent and a uniform quantizer with $b$ quantization bits is employed for recording, we prove that $2^b = \omega\left(\sqrt{m}\right)$ and a quantizer span of $\omega\left(\sqrt{\ln m}\right)$ are sufficient for the adversary to avoid detection; that is, the probability of error $P_e$ of the observer is bounded as $P_e \geq \frac{1}{2} - \epsilon$ for any $\epsilon > 0$. Conversely, having $2^b = \mathcal{O}\left(\sqrt{m}\right)$ or a quantizer span of $o\left(\sqrt{\ln m}\right)$ results in detection by the observer with high probability as $m \to \infty$.**

## I. Introduction

In many scenarios, it is important to know whether a received signal was sent by a friend directly, or whether it was recorded by an adversary and then replayed. This is the common replay attack (or playback attack) in network security, where the attacker records messages from a transmitter to a receiver and replays the massages later to trick the receiver into unauthorized operations. This type of attack can be used effectively in many real world applications like remote keyless-entry system for vehicles [1][2] or text-dependent speaker verification [3].

Similar attacks also occur in radar jamming and deception to protect targets from being detected by enemy radar systems. Deceptive jamming uses techniques like range gate pull-off (RGPO) [4] to break a radar lock from the target. The basic idea is to generate a signal pulse very similar to the one that is reflected by the target, and then send it a fraction of time later so that the radar's range gate starts to follow the false pulse instead of the real reflection. Along with the appearance of digital radio frequency memory (DRFM) [5], a modern deception jammer can capture and retransmit the radiation signal of the target, producing a false signal that confuses the receiver radar and hides the target's real position or velocity.

To prevent a replay attack in network security or deception jamming in radar systems, many approaches have been studied in prior work: [6] presents a comparison of different feature extraction techniques and classifiers for replay attack detection; [7] provides a mobile payment scheme based on radio frequency identification that can prevent replay attack; [8] studies the detection and classification of jamming signals theoretically by analyzing the adaptive coherent estimator and the generalized likelihood ratio test; and [9] proposes a DRFM deception jamming detection approach based on singular spectrum analysis. Although these methods are proposed to efficiently detect false signals that are recorded and replayed in the field of network security and radar jamming, the fundamental limits of such attacks with hardware imperfections has not been explored. In either the replay attack or deception jamming, imperfect hardware such as quantization or nonlinearities of RF components has a significant impact on the detection of signals. Here, motivated by our previous work in identifying transmitters based on subtle imperfections [10][11][12], we initiate a study in employing such imperfections in detecting recording of the signal. Learning the fundamental thresholds for the characteristics of the hardware will provide us with both theoretical insight and application utility.

We start by studying the theoretical limits in the detection of quantized signals. Analogous to work in [13] where a theoretical limit on the amount of information transmitted reliably without detection is presented, we are interested in finding the characteristics of the quantizer employed in a replay system that avoid its detection. Based on the mathematics of statistical hypothesis testing, we provide a limit on the quantization bits and the quantizer span such that the quantized signal essentially cannot be detected. We consider a discrete-time model where the signals are discrete-time series, and provide a brief discussion on the mapping to a continuous-time model later in the paper. If a signal with length $m$ is sent, and the quantization is uniform with $b$ bits, then $2^b = \omega\left(\sqrt{m}\right)$ and a quantizer span of $\omega\left(\sqrt{\ln m}\right)$ are sufficient to avoid detection. Conversely, having $2^b = \mathcal{O}\left(\sqrt{m}\right)$ or a quantizer span of $o\left(\sqrt{\ln m}\right)$ results in detection by the observer with high probability as $m \to \infty$.

The rest of the paper is organized as follows: after introducing the system model and metric in Section II, we derive the optimal hypothesis test and the probability of error in distinguishing the original signal and the quantized signal in Section III. We prove the achievability and converse results in Section IV and Section V respectively. Section VI discuss the mapping to the continuous-time model. Section VII draws the conclusion.

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## II. SYSTEM MODEL AND METRICS

We employ a discrete-time model with real-valued elements and defer discussion of the mapping to a continuous-time model to Section VI. The framework of our system is shown in Fig. 1, where Alice sends a vector $\boldsymbol{X} = \{X_i\}_{i=1}^m$ of $m$ real values and an observer receives a vector $\boldsymbol{Y} = \{Y_i\}_{i=1}^m$. If the signal $\boldsymbol{X}$ is sent directly, which we term as the "original signal", then $Y_i = X_i + N_i$ for $i = 1, 2, \ldots, m$, is an independent and identically distributed (i.i.d.) sequence of Gaussian random variables, where $N_i \sim \mathcal{N}(0, \sigma^2)$ is the noise on the channel between the observer and Alice. However, if the signal is recorded and replayed, then the signal $\boldsymbol{X}$ is first quantized before being sent through the channel; in this case, $Y_i = Q(X_i) + N_i$ for $i = 1, 2, \ldots, m$, where $Q(\cdot)$ is the quantization function with input of the original signal and output of the quantized signal.

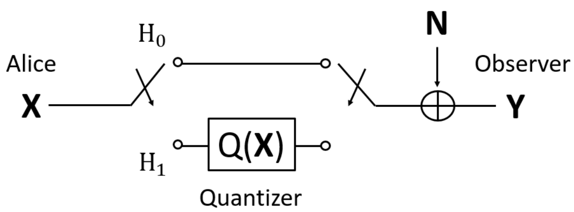


Fig. 1: System framework: Alice sends a real-valued vector $\boldsymbol{X}$ and an observer attempts to classify his observed vector $\boldsymbol{Y}$ as either a vector $\boldsymbol{X} + \boldsymbol{N}$ of the original signal through the channel or a vector $Q(\boldsymbol{X}) + \boldsymbol{N}$ of the quantized signal through the channel.

The goal of our paper is to study how the parameters of the quantizer affect the ability of the observer to distinguish the original signal and the quantized signal. In order to distinguish the two signals, the observer performs a statistical hypothesis test on his observed vector $\boldsymbol{Y}$. We define:

- $H_0$: $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{N}$.
- $H_1$: $\boldsymbol{Y} = Q(\boldsymbol{X}) + \boldsymbol{N}$.

Our metric is the probability of error. We assume that the observer uses classical hypothesis testing with equal prior probabilities of each of the two hypotheses being true. The rejection of $H_0$ when it is true is known as a type I error (or false alarm), and we denote its probability as $\alpha$ [14]. The acceptance of $H_0$ when it is not true is known as a type II error (or miss detection), and we denote $\beta$ to be its probability. Thus, the error probability for the observer to distinguish the two hypotheses can be written as $P_e = \frac{\alpha+\beta}{2}$.

We also assume that the observer knows $\boldsymbol{X}$ and the realization of the quantization function $Q$, and he also knows the variance $\sigma^2$ of the Gaussian noise on the channel between the observer and Alice. So, the observer is aware of the statistics of the two hypotheses $H_0$ and $H_1$.

## III. OPTIMAL TEST AND THE PROBABILITY OF ERROR

The observer's goal is to determine whether his observed vector is the original signal sent through the channel or the signal that is quantized and then sent through the channel. The observer will make his decision based on the optimal test between $H_0$ and $H_1$. The likelihood ratio test (LRT) is:

$$\Omega(\boldsymbol{Y} = \boldsymbol{y}) = \frac{f_{\boldsymbol{Y}|H_0}(\boldsymbol{y})}{f_{\boldsymbol{Y}|H_1}(\boldsymbol{y})} \underset{H_1}{\overset{H_0}{\gtrless}} 1$$

where $f_{\boldsymbol{Y}|H_0}(\boldsymbol{y})$ is the probability distribution of $\boldsymbol{Y}$ given $H_0$ is true and $f_{\boldsymbol{Y}|H_1}(\boldsymbol{y})$ is the probability distribution of $\boldsymbol{Y}$ given $H_1$ is true.

Given that $\boldsymbol{X} = \boldsymbol{x}$, since the observer knows $\boldsymbol{x}$ and $Q(\boldsymbol{x})$, we have $Y_i \sim \mathcal{N}(x_i, \sigma^2), i = 1, 2, \ldots, m$ when $H_0$ is true (i.e., $Y_i = x_i + N_i$), and $Y_i \sim \mathcal{N}(Q(x_i), \sigma^2), i = 1, 2, \ldots, m$ when $H_1$ is true (i.e., $Y_i = Q(x_i) + N_i$). Thus, the LRT can be written as:

$$\Omega(\boldsymbol{Y} = \boldsymbol{y}) = \frac{\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i - x_i)^2}{\sigma^2}}}{\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i - Q(x_i))^2}{\sigma^2}}} \underset{H_1}{\overset{H_0}{\gtrless}} 1$$

which could be further written as:

$$\sum_{i=1}^{m} \left[ -(y_i - x_i)^2 + (y_i - Q(x_i))^2 \right] \underset{H_1}{\overset{H_0}{\gtrless}} 1$$

i.e.,

$$\sum_{i=1}^{m} \left[ 2y_i(x_i - Q(x_i)) + (Q(x_i))^2 - x_i^2 \right] \underset{H_1}{\overset{H_0}{\gtrless}} 1$$

Now we would like to derive the error probability for the above test. Let us denote $Z_i = 2Y_i(X_i - Q(X_i)) + (Q(X_i))^2 - X_i^2, i = 1, 2, \ldots, m$. Given $H_0$ is true and $\boldsymbol{X}$ is known to the observer, then $Y_i = X_i + N_i$ and $Y_i \sim \mathcal{N}(X_i, \sigma^2), i = 1, 2, \ldots, m$. In this case, we see that $Z_i$ is a Gaussian random variable with mean $2X_i(X_i - Q(X_i)) + (Q(X_i))^2 - X_i^2$ and variance $4(X_i - Q(X_i))^2\sigma^2$. Thus, under the assumption of equal prior probabilities of $H_0$ and $H_1$, the probability of false alarm is given by:

$$\alpha = P\left(\sum_{i=1}^{m} Z_i < 0\right) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{\sum_{i=1}^{m} U_i^2}}{2\sqrt{2}\sigma}\right)$$

where $U_i = X_i - Q(X_i)$, $i = 1, 2, \ldots, m$, is the quantization error which is uniformly distributed over $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$. The quantization step size is denoted as $\Delta$.

By the symmetry of the problem, the probability of miss detection has the same form ($\alpha = \beta$). Thus, the error probability for the observer to distinguish the two hypotheses is given by:

$$P_e(\boldsymbol{X}) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{\sum_{i=1}^{m} U_i^2}}{2\sqrt{2}\sigma}\right) \tag{1}$$

In order to confuse the observer, we want the error probability $P_e$ to be close to $\frac{1}{2}$ so that it is equivalent to a random guess for the observer in distinguishing the original signal and the quantized signal, meaning that we want $P_e \geq \frac{1}{2} - \epsilon$ for any $\epsilon > 0$ [13]. Conversely, we want $P_e \leq \epsilon$ for any $\epsilon > 0$ for the observer to be able to detect the quantized signal with high probability.

## IV. ACHIEVABILITY

In this section, we will state the achievability theorems under the assumption that the quantizer in our system is uniform. We seek the sufficient step size $\Delta$ as a function of the vector length $m$. Intuitively this $\Delta$ is a small number, and since we do not want $\Delta \to \infty$, we must have $\Delta = \mathcal{O}(1)$ for all circumstances. We can write $\Delta = \frac{C}{2^b}$ where $b$ is the number of quantization bits and $C$ is a constant.

**Theorem 1** (Achievability under uniform quantization)**.** *Suppose that Alice sends a discrete-time signal with length $m$ and the quantizer is uniform with $b$ bits of quantization. If $2^b = \omega(\sqrt{m})$ (in particular, $b \geq \log_2 \frac{C\sqrt{m}}{4\sqrt{6}\sigma \mathrm{erf}^{-1}(2\epsilon)}$, where $C$ is a constant and $\sigma$ is the standard deviation of the noise on the channel), then the observer can only distinguish the original signal and the quantized signal with error probability $P_e \geq \frac{1}{2} - \epsilon$ for any $\epsilon > 0$.*

*Proof.* We will first take the expectation of $P_e(\boldsymbol{X})$ in (1) over $U_i \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$ for $i = 1, 2, \ldots, m$. Since $-\mathrm{erf}(\cdot)$ is a convex function, then by Jensen's inequality we have:

$$P_e = E_{\boldsymbol{X}}[P_e(\boldsymbol{X})] \geq \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{E[\sum_{i=1}^{m} U_i^2]}}{2\sqrt{2}\sigma}\right) \tag{2}$$

which is a tight lower bound since $\mathrm{erf}(x)$ is approximately linear at small $x$. Substituting $E[\sum_{i=1}^{m} U_i^2] = \frac{m\Delta^2}{12}$ and $\Delta = \frac{C}{2^b}$ in (2) yields:

$$P_e \geq \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\frac{C}{2^b}\sqrt{\frac{m}{12}}}{2\sqrt{2}\sigma}\right)$$

If $b \geq \log_2 \frac{C\sqrt{m}}{4\sqrt{6}\sigma \mathrm{erf}^{-1}(2\epsilon)}$ for any $\epsilon > 0$, then $\frac{1}{2}\mathrm{erf}\left(\frac{\frac{C}{2^b}\sqrt{\frac{m}{12}}}{2\sqrt{2}\sigma}\right) \leq \epsilon$. This implies that $P_e \geq \frac{1}{2} - \epsilon$ for any $\epsilon > 0$. Thus, $2^b = \omega(\sqrt{m})$ is sufficient to prevent detection by the observer. □

In many scenarios, the transmitted signals can have a wide range of values; for example, a Gaussian signal would have values ranging from negative infinity to infinity. In this case, the quantizer would have overflows that can assist the observer in detecting the signal. So, we next assume that the original signal $X_i \sim \mathcal{N}(0, \sigma_0^2), i = 1, 2, \ldots, m$. We consider that the system employs quantization with a span of $[-l, l]$. If its input has value within this span, it outputs the quantized value using the quantization function $Q$. If the input value is outside of the span, it overflows and outputs either $-l$ or $l$. To keep the observer from detecting the overflows, $l$ should be scaled with the length $m$ of the transmitted signal. Intuitively, $l$ must go to infinity as $m \to \infty$ or the quantization is readily detected. Thus, we consider $l = \omega(1)$ for all circumstances. We obtain the achievability result in this case as below.

**Theorem 2** (Achievability under uniform quantization with overflows)**.** *Suppose that Alice sends a discrete-time signal with length $m$ and the quantizer is uniform with $b$ bits of quantization and a span of $[-l, l]$. Then, if $2^b = \omega(\sqrt{m})$ bits and $l = \omega\left(\sqrt{\ln m}\right)$, the observer can only distinguish the original signal and the quantized signal with error probability $P_e \geq \frac{1}{2} - \epsilon$ for any $\epsilon > 0$.*

*Proof.* Since the quantizer has a span of $[-l, l]$, then if hypothesis $H_1$ is true, $Y_i$ can be written as:

$$Y_i = \begin{cases} Q(X_i), & |X_i| < l \\ l, & X_i > l \\ -l, & X_i < -l \end{cases}, \quad i = 1, 2, \ldots, m$$

and the quantization error can be written as:

$$U_i = \begin{cases} X_i - Q(X_i), & |X_i| < l \\ X_i - l, & X_i > l \\ X_i + l, & X_i < -l \end{cases}, \quad i = 1, 2, \ldots, m \tag{3}$$

Recall that $X_i \sim \mathcal{N}(0, \sigma_0^2)$ and $E[U_i^2] = \frac{\Delta^2}{12}$ when $U_i = X_i - Q(X_i)$ for $i = 1, 2, \ldots, m$. Then, we derive the expectation of $U_i^2$ for the general case in (3) as:

$$\begin{aligned} E[U_i^2] =& P(|X_i| < l) \cdot \frac{\Delta^2}{12} \\ &+ P(X_i > l) \cdot \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x+l)^2}{2\sigma_0^2}} x^2 dx \\ &+ P(X_i < -l) \cdot \int_{-\infty}^0 \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x-l)^2}{2\sigma_0^2}} x^2 dx \\ =& \frac{\Delta^2}{12}\mathrm{erf}\left(\frac{l}{\sqrt{2}\sigma_0}\right) \\ &+ \left(\frac{\sigma_0^2 l e^{-\frac{l^2}{2\sigma_0^2}}}{\sqrt{2\pi}} + \frac{l^2 + \sigma_0^2}{2}\left(1 - \mathrm{erf}\left(\frac{l}{\sqrt{2}\sigma_0}\right)\right)\right) \\ &\cdot \left(1 - \mathrm{erf}\left(\frac{l}{\sqrt{2}\sigma_0}\right)\right) \end{aligned} \tag{4}$$

Substituting (4) in (2) we have a lower bound for $P_e$ in this case when the quantizer has overflows:

$$P_e \geq \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{mE[U_i^2]}}{2\sqrt{2}\sigma}\right) \tag{5}$$

Note that $l$ goes to infinity as $m \to \infty$. If $2^b = \omega(\sqrt{m})$, then $\frac{m\Delta^2}{12}\mathrm{erf}\left(\frac{l}{\sqrt{2}\sigma_0}\right) \leq \frac{m\Delta^2}{12} \to 0$, i.e., the first term of $mE[U_i^2]$ goes to zero as $m \to \infty$. On the other hand, if

$l=\omega\left(\sqrt{\ln m}\right)$, we have:

$$m\left(\frac{\sigma_0^2 l e^{-\frac{l^2}{2\sigma_0^2}}}{\sqrt{2\pi}}+\frac{l^2+\sigma_0^2}{2}\cdot\frac{e^{-\frac{l^2}{2\sigma_0^2}}}{\sqrt{\pi}\frac{l}{\sqrt{2}\sigma_0}}\right)\frac{e^{-\frac{l^2}{2\sigma_0^2}}}{\sqrt{\pi}\frac{l}{\sqrt{2}\sigma_0}}\to 0 \quad (6)$$

since keeping only the dominant terms in (6) yields $\frac{m\sigma_0(\sigma_0^2+1)e^{-\frac{l^2}{\sigma_0^2}}}{\pi}\to 0$ when $l=\omega\left(\sqrt{\ln m}\right)$. If we take the Taylor series expansion of $\operatorname{erf}\left(\frac{l}{\sqrt{2}\sigma_0}\right)$ at $\frac{l}{\sqrt{2}\sigma_0}=\infty$, then the second term of $mE[U_i^2]$ is upper bounded by (6), and hence goes to zero as $m\to\infty$.

Therefore, letting $2^b=\omega(\sqrt{m})$ and $l=\omega\left(\sqrt{\ln m}\right)$, we get $mE[U_i^2]\to 0$ as $m\to\infty$, which implies that $\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{mE[U_i^2]}}{2\sqrt{2}\sigma}\right)\le\epsilon$ for any $\epsilon>0$. By (5), we have $P_e\ge\frac{1}{2}-\epsilon$ for any $\epsilon>0$.

□

## V. CONVERSE

In this section, we provide the converse results under the assumption that the quantizer is uniform with step size $\Delta$.

**Theorem 3** (Converse under uniform quantization)**.** *Suppose that Alice sends a discrete-time signal with length $m$ and the quantizer is uniform with $b$ bits of quantization levels. If $2^b=\mathcal{O}(\sqrt{m})$ (in particular, $b\le\log_2\frac{C\sqrt{m}}{8\sqrt{2}\sigma\sqrt{\ln\frac{1}{2\epsilon}}}$ for any $\epsilon>0$, where C is a constant and $\sigma$ is the standard deviation of the noise on the channel), then the observer can distinguish the original signal and the quantized signal with arbitrarily low probability of error.*

*Proof.* For the achievability result, we give a tight lower bound (2) to the error probability $P_e$. Now, we need an analogous upper bound. Using the fact that $\operatorname{erf}(x)\ge 1-e^{-x^2}$ [15], we upper bound $P_e(\boldsymbol{X})$ in (1) as:

$$P_e(\boldsymbol{X})\le\frac{1}{2}e^{-\frac{\sum_{i=1}^{m}U_i^2}{8\sigma^2}}=\frac{1}{2}\prod_{i=1}^{m}e^{-\frac{U_i^2}{8\sigma^2}} \quad (7)$$

and taking the expectation yields:

$$\begin{aligned}P_e=E_{\boldsymbol{X}}[P_e(\boldsymbol{X})]&\le\frac{1}{2}\left(\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}e^{-\frac{x^2}{8\sigma^2}}\frac{1}{\Delta}dx\right)^m\\&=\frac{1}{2}\left(\frac{2\sqrt{2\pi}\sigma}{\Delta}\operatorname{erf}\left(\frac{\Delta}{4\sqrt{2}\sigma}\right)\right)^m \quad (8)\end{aligned}$$

If $2^b=\mathcal{O}(\sqrt{m})$, $\Delta$ is small, and thus we take the Taylor series expansion of $\operatorname{erf}\left(\frac{\Delta}{4\sqrt{2}\sigma}\right)$ at $\frac{\Delta}{4\sqrt{2}\sigma}=0$:

$$\begin{aligned}P_e&\le\frac{1}{2}\left(\frac{4\sqrt{2}\sigma}{\Delta}\left(\frac{\Delta}{4\sqrt{2}\sigma}-\frac{\Delta^3}{3(4\sqrt{2}\sigma)^2}+\frac{\Delta^5}{10(4\sqrt{2}\sigma)^5}\right)\right)^m\\&=\frac{1}{2}\left(1-\frac{\Delta^2}{3(4\sqrt{2}\sigma)^2}+\frac{\Delta^4}{10(4\sqrt{2}\sigma)^4}\right)^m\end{aligned}$$

Note that $e^{-\frac{\Delta^2}{4(4\sqrt{2}\sigma)^2}}=1-\frac{\Delta^2}{4(4\sqrt{2}\sigma)^2}+\frac{\Delta^4}{32(4\sqrt{2}\sigma)^4}+\mathcal{O}(\Delta^5)$ at $\frac{\Delta}{4\sqrt{2}\sigma}=0$. When $\Delta$ is small, the first and second terms are dominant. Thus, we have:

$$P_e\le\frac{1}{2}e^{-\frac{m\Delta^2}{4(4\sqrt{2}\sigma)^2}}+\mathcal{O}\left(\Delta^3\right) \quad (9)$$

For large $m$, we can ignore the error term $\mathcal{O}\left(\Delta^3\right)$. Then, if $b\le\log_2\frac{C\sqrt{m}}{8\sqrt{2}\sigma\sqrt{\ln\frac{1}{2\epsilon}}}$ for any $\epsilon>0$, $P_e\le\epsilon$ for any $\epsilon>0$. Thus, $2^b=\mathcal{O}(\sqrt{m})$ is sufficient for the observer to detect the signal. □

Now we consider the case that the system employs a quantizer with overflows; in particular, it has a span of $[-l,l]$, and we assume that the original signal $X_i\sim\mathcal{N}(0,\sigma_0^2), i=1,2,\ldots,m$. We provide the converse result in this case.

**Theorem 4** (Converse under uniform quantization with overflows)**.** *Suppose that Alice sends a discrete-time signal with length $m$ and the quantizer is uniform with $b$ bits of quantization levels. Then, if $b=\mathcal{O}(\sqrt{m})$ or $l=o\left(\sqrt{\ln m}\right)$, the observer can distinguish the original signal and the quantized signal with arbitrarily low probability of error.*

*Proof.* We derive the upper bound for $P_e$ in the case that the quantizer has overflows. In this case, the quantization error $U_i$ for $i=1,2,\ldots,m$ is given by (3). Following (7), we write:

$$\begin{aligned}P_e=E_{\boldsymbol{X}}[P_e(\boldsymbol{X})]\le\frac{1}{2}\Bigg[&P(|X_i|<l)\cdot\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}e^{-\frac{x^2}{8\sigma^2}}\frac{1}{\Delta}dx\\&+P(X_i>l)\cdot\int_0^{\infty}\frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{(x+l)^2}{2\sigma_0^2}}e^{-\frac{x^2}{8\sigma^2}}dx\\&+P(X_i<-l)\cdot\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{(x-l)^2}{2\sigma_0^2}}e^{-\frac{x^2}{8\sigma^2}}dx\Bigg]^m\\=\frac{1}{2}\Bigg[&\underbrace{\operatorname{erf}\left(\frac{l}{\sqrt{2}\sigma_0}\right)\frac{2\sqrt{2\pi}\sigma}{\Delta}\operatorname{erf}\left(\frac{\Delta}{4\sqrt{2}\sigma}\right)}_{A}+\\&\underbrace{\frac{\left(1-\operatorname{erf}\left(\frac{l}{\sqrt{2}\sigma_0}\right)\right)e^{-\frac{l^2}{8\sigma^2+2\sigma_0^2}}\left(1-\operatorname{erf}\left(\frac{\sqrt{2}l}{\sqrt{\frac{\sigma_0^4}{\sigma^2}+4\sigma_0^2}}\right)\right)}{\sqrt{\frac{\sigma_0^2}{\sigma^2}+4}}}_{B}\Bigg]^m\end{aligned}$$

If $2^b=\mathcal{O}(\sqrt{m})$ and $l$ is arbitrary, then by the discussion from (8) to (9), we have:

$$\begin{aligned}A^m&\le\left(\frac{2\sqrt{2\pi}\sigma}{\Delta}\operatorname{erf}\left(\frac{\Delta}{4\sqrt{2}\sigma}\right)\right)^m\\&\le e^{-\frac{m\Delta^2}{4(4\sqrt{2}\sigma)^2}}+\mathcal{O}\left(\Delta^3\right) \quad (10)\end{aligned}$$

which goes to zero as $m\to\infty$. On the other hand, recall that $l=\omega(1)$, then for any $a\ge 1$, $B^a\to 0$ as $l$ becomes large. This can be seen by noting that if we ignore all of the constants in $B$ and use the fact that $1 - \text{erf}(x) < e^{-x^2}$ [15], we have $B < e^{-3l^2}$. Thus, if $2^b = \mathcal{O}(\sqrt{m})$, $P_e \leq \epsilon$ for any $\epsilon > 0$.

If $l = o\left(\sqrt{\ln m}\right)$ and $b$ is arbitrary, then $A^m \leq \left(\text{erf}\left(\frac{l}{\sqrt{2}\sigma_0}\right)\right)^m \to 0$ as $m \to \infty$. Again, for any $a \geq 1$, $B^a \to 0$ as $l$ becomes large. Thus, when $l = o\left(\sqrt{\ln m}\right)$, $P_e \leq \epsilon$ for any $\epsilon > 0$.

Therefore, if $2^b = \mathcal{O}(\sqrt{m})$ or $l = o\left(\sqrt{\ln m}\right)$, the error probability $P_e$ at the observer is arbitrarily small, which establishes the converse result. $\square$

## VI. Discussion

For this paper, we employ a discrete-time model with real-valued signals. However, although this is commonly assumed without loss of generality in standard communication theory, it is worthwhile to consider whether it is sufficient to focus on discrete time here. We provide a brief discussion on the mapping to a continuous-time model.

Consider the standard communication system model, where Alice transmits a waveform:

$$X(t) = \sum_{i=-\infty}^{\infty} X_i \cdot p(t - iT_s)$$

where $T_s$ is the symbol period and $p(\cdot)$ is the pulse shaping filter. If the target uses ideal pulse shaping, i.e., $p(x) = \text{sinc}\left(\frac{x}{T_s}\right) = \frac{\sin\left(\frac{\pi x}{T_s}\right)}{\frac{\pi x}{T_s}}$, it is sufficient for optimal detection for the observer to sample at a rate of $\frac{1}{T_s}$ samples/second, which leads directly to the discrete-time model. However, $p(\cdot)$ would generally be chosen more practically, such as a raised-cosine filter with configurable excess bandwidth. In this case, the observer would sample at a rate higher than $\frac{1}{T_s}$ to obtain all the information. Such a scenario is left for future work.

## VII. Conclusion

In many applications such as the detection of a replay attack in network security or the detection of deception jamming in radar systems, it is important to know whether a received signal was sent directly, or was recorded and then replayed. Many approaches to this problem have been proposed in prior work; however, the fundamental limits of such detection with hardware imperfections have not been explored. Thus, we have studied this limit and analyzed the characteristics of the hardware, in particular the quantizer, that affect the detection. Specifically, if a signal with length $m$ is sent and a uniform $b$-bit quantizer is employed, then $2^b = \omega(\sqrt{m})$ and a quantizer span of $\omega\left(\sqrt{\ln m}\right)$ are sufficient to avoid detection; that is, the error probability at the observer is bounded as $P_e \geq \frac{1}{2} - \epsilon$ for any $\epsilon > 0$. Conversely, having $2^b = \mathcal{O}(\sqrt{m})$ or a quantizer span of $o\left(\sqrt{\ln m}\right)$ results in detection by the observer with high probability as $m \to \infty$. Although we considered a discrete-time model, it can be directly mapped to a continuous-time model if we use a perfect pulse shaping filter. Scenarios with more practical choices of pulse shaping filters provide a promising area for future work.

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