

Landauer's conductance formula and its generalization to finite voltages

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We recast the expression for tunneling current to show that the energy averaging due to a finite voltage is statistically independent from the energy averaging due to finite temperature. The energy averaging takes the mathematical form of convolutions. Convolving the standard Landauer conductance formula in energy with a voltage windowing function averages the conductance over an energy range equal to the applied voltage, just as the independent convolution with the Fermi-Dirac probability density function averages the conductance over an energy range equal to the temperature. We illustrate the effects of voltage broadening versus ordinary thermal broadening of the conductance using a quasi-one-dimensional ballistic conductor as a model system, as in the recent experiments of van Wees *et al.* [Phys. Rev. Lett. **60**, 848 (1988)] and Wharam *et al.* [J. Phys. C **21**, L209 (1988)].

I. INTRODUCTION

Since Landauer's original work^{1,2} relating electrical conductance to the transmission probability through a region of elastic scatterers between two temperature baths, his formula has been extended to the case of multiple quantum channels,^{3,4} extended to include inelastic scattering within the sample,⁵ applied to many different experimental geometries,^{6,7} and derived from linear-response theory.^{4,8} A review of the two types of Landauer formulas, discussing the different measurement conditions under which each applies, is given in Refs. 6, 8, 9, and 10. References 6, 8, and 10 review the higher-dimensional analogs of these two types of Landauer formulas. The extension of Landauer's formula to finite voltages has also been investigated.^{11,12}

The main goal of this paper is to express the effects of energy averaging due to finite voltage and finite temperature on the electrical conductance mathematically in terms of convolutions. This objective is important because convolutions are a very natural way to incorporate random processes leading to a broadening of energy levels, as is the case with the energy level broadening due to elastic and inelastic scattering in a normal dirty metal.¹³ We show in Sec. II that to generalize the zero-voltage and zero-temperature Landauer conductance formula to finite voltage and finite temperature, one merely convolves it in energy with both a voltage windowing function and a thermal smearing function, respectively. These two convolutions show that the effect of finite temperature is to average the conductance over a region $k_B T$ near the Fermi level, while one effect of finite voltage is to average the conductance over an energy range eV near the Fermi level, as previously mentioned in Refs. 11 and 12. By proving that the difference in Fermi functions characterizing electrical conduction can be written as a convolution of

two separate functions, one function depending only on the applied voltage and the other depending only on temperature, we show in this paper that voltage broadening and temperature broadening are statistically independent. We apply this convolution broadening method to a quasi-one-dimensional ballistic conductor to understand the qualitative similarities and differences between voltage broadening and ordinary thermal broadening of the conductance. By emphasizing that electrical conduction necessarily occurs at finite voltages, we gain additional insight into the physical mechanism responsible for the quantum contact resistance in one-dimensional ballistic conductors. In Sec. III we examine Landauer's formula in the classical diffusive limit in one dimension, showing that it reduces to the standard Drude conductance result. We show that one also obtains the quantum contact resistance from the Drude-Sommerfeld conductance formula in the limit that the mean free path grows to its upper bound of the device size.

A subsidiary goal of this paper is to examine the relationship of Landauer's conductance formula to previous treatments of electron tunneling at finite voltages. In the tunneling limit, by which we mean many parallel quantum channels with no scattering between them, we show in Sec. IV that the finite-voltage Landauer formula reduces to many well-known treatments of electron tunneling at finite voltages.^{12,14,15} However, even in this well-known tunneling limit, we can gain physical insight into the shape of the current density versus Fermi energy in different spatial dimensions by noting that, because the tunneling Hamiltonian is separable, we can obtain the tunneling current density in two spatial dimensions by convolving the one-dimensional tunneling current with the one-dimensional free-electron density of states, repeating this process to obtain the three-dimensional tunneling current density.

II. LANDAUER'S FORMULA IN ONE DIMENSION

Our model for a quantum conductor in one dimension is a potential $U(x, V)$ between two thermal reservoirs as in Fig. 1, where U depends both on position x and the applied voltage V . The potential $U(x, V)$ depends on the applied voltage because electrons reflected from a scattering obstacle pile up on one side of the obstacle, as emphasized in Ref. 1, leading to a density gradient and nonuniform electric field concentrated around the obstacle. These density and potential gradients in turn influence the tunneling potential so that $U(x, V)$ must be determined self-consistently, and our treatment totally neglects such effects. We emphasize that the self-consistent determination of $U(x, V)$ is the central issue in any specific calculation of tunneling currents. An approximate method to self-consistently determine the tunneling potential at finite voltages, but which still assumes a uniform electric field, is given in Ref. 11. In this paper we will seek results which will not depend strongly on the exact shape of the tunneling potential, as long as the electron tunnels through some $U(x, V)$.

Electrons deep inside the two reservoirs are assumed to have a Fermi-Dirac distribution shown in Fig. 1. We assume the effect of an applied voltage between the two reservoirs is to create an imbalance in the electrochemical potentials deep inside the reservoirs equal to the applied voltage. This results in a current¹⁴

$$I(\mu, V, T) = e \int v^+(E) T(E, V) N^+(E) \times [f(E - (\mu + eV), T) - f(E - \mu, T)] dE, \quad (1)$$

where $v^+(E)$ is the group velocity for electrons moving in the positive x direction, $N^+(E)$ the density of states for electrons of both spins moving in the positive x direction in a one-dimensional free-electron gas, f is the Fermi function, T the temperature of both reservoirs, and

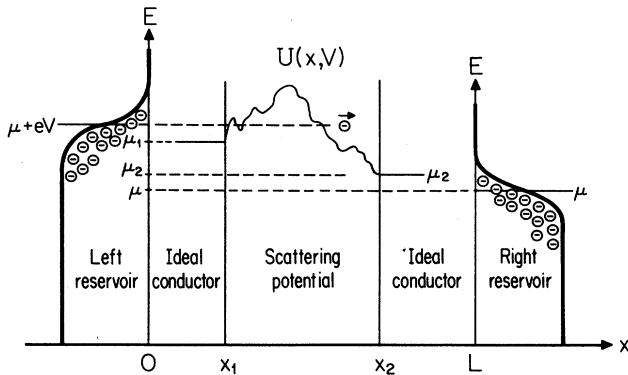


FIG. 1. Model for a one-dimensional quantum conductor: An elastic scattering potential $U(x, V)$ located between two thermal reservoirs. μ_1 and μ_2 are the electrochemical potentials in the left and right ideal conductors, respectively. An applied voltage V creates an imbalance between the electrochemical potentials $\mu_l = \mu + eV$ in the left reservoir and $\mu_r = \mu$ in the right reservoir. Note that $\mu_l \neq \mu_1$ and $\mu_r \neq \mu_2$.

$T(E, V)$ the transmission coefficient through the potential $U(x, V)$. By enforcing time reversal symmetry and current conservation during tunneling, it is possible to show that the current transmission probability through the potential $U(x, V)$ is the same from left to right as from right to left, i.e., $T(E, V) = T_r(E, V) = T_l(E, V)$. While Eq. (1) is intuitive, it is also possible to derive it using the transfer Hamiltonian method and Fermi's golden rule.^{12,14}

The product of the group velocity and the electron density of states in one dimension is a constant given by

$$v^+(E) N^+(E) = \left[\frac{1}{\hbar} \left| \frac{dE}{dk} \right| \right] \left[\frac{1}{\pi} \frac{1}{|dE/dk|} \right] = \frac{1}{\pi \hbar}. \quad (2)$$

The Fermi function in Eq. (1) can itself be expressed as a composite probability function

$$f(E - \mu, T) = [1 - \theta(E - \mu)] \otimes \left[-\frac{df}{dE}(E, T) \right], \quad (3)$$

where \otimes denotes a convolution in energy, θ is the unit step function, and

$$-\frac{df}{dE}(E, T) = \frac{1}{4k_B T} \operatorname{sech}^2 \left[\frac{E}{2k_B T} \right] \quad (4)$$

is the thermal smearing function or Fermi-Dirac probability density function. The convolution \otimes of two functions $A(E)$ and $B(E)$ has its usual meaning,

$$A(E) \otimes B(E) = \int_{-\infty}^{\infty} A(E - E') B(E') dE' = \int_{-\infty}^{\infty} A(E') B(E - E') dE'. \quad (5)$$

The difference of Fermi functions in Eq. (1) can therefore be written as

$$f(E - (\mu + eV), T) - f(E - \mu, T) = \{ \theta(E - \mu) - \theta[E - (\mu + eV)] \} \otimes \left[-\frac{df}{dE}(E, T) \right]. \quad (6)$$

Equation (6) shows that the difference of Fermi functions in Eq. (1) can be expressed as the convolution in energy of two independent functions, one of which depends only on the applied voltage and the other which depends only on the temperature. Equation (6) therefore separates the effect of finite voltage on the electrical current from the effect of finite temperature.

Our expression for the current now becomes

$$I(E, V, T) = \frac{e}{\pi \hbar} T(E, V) \otimes W(E, V) \otimes \left[-\frac{df}{dE}(E, T) \right] \equiv \frac{e^2 V}{\pi \hbar} \bar{T}(E, V). \quad (7)$$

Equation (7), where $W(E, V) = [\theta(E + eV) - \theta(E)]$, is our interpretation of the finite-voltage Landauer formula in one dimension in terms of convolutions. $\bar{T}(E, V)$ is the energy averaged transmission coefficient. After carrying out the convolutions we set $E = \mu$. We display Eq. (7)

graphically in Fig. 2. At this point we ignore the last two convolutions shown in the figure. If both eV and $k_B T$ are small compared with any structure in the transmission coefficient $T(E, V)$, Eq. (7) reduces to the well-known result

$$I(E) = \frac{e^2}{\pi\hbar} T(E) V = \frac{V}{R_t} \quad (8)$$

which is Landauer's conductance formula in one dimension. Here R_t is the total resistance of the electron moving from one contact to the other. This formula was also obtained explicitly in Ref. 12 in a discussion of tunneling in one dimension between two thermal reservoirs.

Equation (7) has a simple physical interpretation: The convolution with the voltage window $W(E, V) = [\theta(E + eV) - \theta(E)]$ implies that the electron beam near the Fermi surface contributing to conduction is not monochromatic; it has a width in energy of eV . The box-like shape of $W(E, V)$ as shown in Fig. 2 argues that the electron beam near the Fermi surface has a uniform probability to have any energy between the electrochemical potential in the right contact and that energy plus eV . Equivalently, the average tunneling electron near the Fermi energy has a variance about its mean energy of $\Delta E \simeq eV$ due to the applied voltage.¹⁶ The convolution with $-df/dE$ is just the standard thermal broadening of the conductance. Just as thermal broadening of the conductance makes it necessary to average the zero-temperature conductance over an energy range $k_B T$ near the Fermi level, finite voltages require averaging the conductance over an energy scale eV near the Fermi level. The separate convolutions imply that thermal broadening and voltage broadening are statistically independent. Note also that the phase coherence time $\tau_V = \hbar/eV$ introduced by the nonmonochromatic electron beam at finite voltages is analogous to the better-known phase coherence time at finite temperatures $\tau_T = \hbar/k_B T$, both times

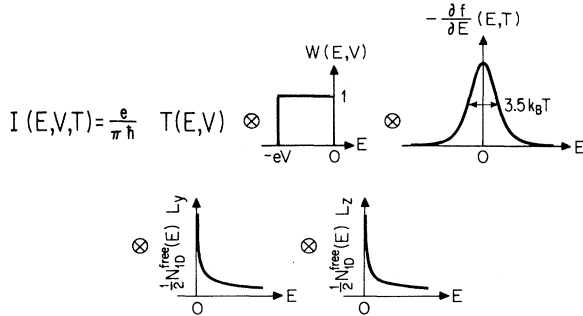


FIG. 2. Effect of finite temperature and finite voltage on the tunneling current displayed graphically as independent convolutions. Energy averaging due to finite voltages is included in the voltage window $W(E, V)$, while the energy averaging due to finite temperature results from the thermal smearing function $-df(E, T)/dE$. The two convolutions with the one-dimensional free-electron density of states relate tunneling current density in three dimensions to the one-dimensional tunneling current.

being set approximately by the variances of $W(E, V)$ and $-df(E, T)/dE$, respectively.

Equation (8) predicts that a conductor having perfect transmission $T(E)=1$ has a contact resistance of $R_c = h/2e^2$. This result seems contrary to one's intuition from the Drude formula as well as from Landauer's original formula,¹⁷ which gives the device resistance R_d as

$$R_d = \frac{h}{2e^2} \frac{R(E)}{T(E)}. \quad (9)$$

Here $R(E)$ is the reflection coefficient so that $T(E) + R(E) = 1$. Büttiker⁶ discusses the assumptions necessary to obtain Eq. (9). Both the Drude formula as well as Landauer's original formula, Eq. (9), assert that a length of perfect conductor has zero resistance.¹⁷ It is now understood^{9,10} that whether one obtains Eq. (8) or (9) in an actual experiment depends on how the resistance is measured. Connecting two weakly coupled voltage probes to the ideal conductors in Fig. 1, which measure the electrochemical potentials μ_1 and μ_2 as defined in Ref. 18, one obtains the original Landauer formula

$$V_d = (\mu_1 - \mu_2)/e = IR_d = I(h/2e^2)(R/T),$$

which is Eq. (9). Landauer's original formula for R_d corresponds to the standard four-probe geometry for measuring resistance, in which the voltmeter is only weakly coupled to the sample one wishes to measure and no current flows into the voltmeter. If, on the other hand, one wishes to obtain the resistance using a two-probe geometry, which measures the electrochemical potentials $\mu + eV = \mu_l$ and $\mu = \mu_r$, one obtains

$$V_l = V = (\mu_l - \mu_r)/e = IR_l = I(h/2e^2)(1/T)$$

which is Eq. (8). In this case the voltmeter is the reservoir, which is strongly coupled to the system one wishes to measure. Due to the incoherence introduced by the reservoirs⁵ in Fig. 1, the total resistance R_t is simply the sum of the device resistance R_d plus the contact resistance R_c , i.e., $R_t = R_d + R_c$. That R_c is in fact a contact resistance associated with the introduction of the two reservoirs in Fig. 1 was explained by Imry¹⁰ in terms of the effusion of a gas through a small hole, and by Büttiker^{9,6} who emphasized that carrier motion from one reservoir to the other was essential to obtain the contact resistance.

The contact resistance implied by Eq. (8) follows from the argument in Eq. (1) that the current is simply a product of the carrier charge, velocity, and density. For small temperatures and voltages this gives

$$I = ev_F^+ n^+(E_F) = ev_F^+ [N^+(E_F)eV] = \frac{e^2}{\pi\hbar} V,$$

where $n^+(E_F) = N^+(E_F)eV$ is the carrier density near the Fermi level. As the Fermi level is raised, the Fermi velocity increases but the carrier density decreases. Equation (2) tells us that, in one dimension, these two factors exactly cancel each other for all energies. The contact resistance in Eq. (8) is then a consequence of the seemingly fortuitous cancellation in Eq. (2). We wish to propose a heuristic explanation for Eq. (8), which does

not rely on Eq. (2), and which emphasizes the way in which current is drawn out at a contact.

To gain further insight into the contact resistance and make a plausibility argument for Eq. (8), introduce the times

$$I = \frac{2e}{\tau_c + \tau_d} = \frac{e^2}{\pi\hbar} T(E)V \text{ and } I = \frac{2e}{\tau_c} = \frac{e^2}{\pi\hbar} V \quad (10)$$

which define the effective time of an electron in the device τ_d and the contact τ_c . One obtains for these times

$$\tau_c = \frac{2\pi\hbar}{eV} \simeq \tau_V \text{ and } \tau_d = \frac{2\pi\hbar}{eV} \frac{1-T(E)}{T(E)}. \quad (11)$$

The time τ_d corresponds to the original Landauer formula R_d while the time τ_c corresponds to the contact resistance R_c . If the conductor is in the classical diffusive limit, the time τ_d is proportional to the time for a classical particle to diffuse across the conductor.¹⁹

We can now make a plausibility argument for Eq. (8). We consider a single quantum ballistic channel having unity transmission so that the resistance R_d is negligible, i.e., $\tau_d \rightarrow 0$. That the conductance of the perfect wire by itself is infinite one can understand simply from the Drude formula, in which an impulse function electric field excites a current which flows forever without decaying. Our emphasis must be on understanding the resistance R_c , which we do by arguing for the time τ_c as follows: Consider the case of zero temperature. Far away from the interface between the device and the reservoir, the right contact must be described by a Fermi function having a single current-carrying state at the electrochemical potential μ_r . As the right contact rapidly draws out an electron at energy μ_r , another electron can occupy that energy level. It cannot, however, do so instantaneously. When the average tunneling electron near the Fermi surface reaches the right contact, it must dissipate on average an amount of energy $\Delta E = eV$ in that contact. Even if inelastic scattering processes occur on very rapid time scales in the contact, there is a limit imposed by the uncertainly principle for the rate at which energy can be dissipated into the thermal bath; namely $1/\Delta\tau = \Delta E/\hbar$. The time $\Delta\tau$ to dissipate an amount of energy $\Delta E = eV$ into the contact should be the same order of magnitude as the time τ_c to equilibrate with the measurement reservoir. This equilibration step will limit the current flow because, if the electron is out of equilibrium with the measurement reservoir, the reservoir cannot detect the electron. Therefore, only one electron of charge e can flow across the device every \hbar/eV sec.²⁰ Imposing this rate-limiting process in the contacts, we argue that the electrical current even in this ballistic case is

$$I \simeq \frac{e}{\Delta\tau} \simeq \frac{e}{\hbar} \Delta E \simeq \frac{e^2}{\hbar} V, \quad (12)$$

which is the same order of magnitude as Eq. (8). This contact resistance corresponds to energy dissipation in the reservoirs, since no energy can be dissipated by elastic scattering in the sample. For the electron to dissipate an amount of energy eV in the reservoirs requires some time $\tau_c = 2\pi\hbar/eV$, which limits the rate at which current can

be transported into the right contact even in a perfectly ballistic device. Since the rate-limiting step for the current occurs in the contact, not in the device itself, the conductance formula (7) is independent of the length of the device as $T(E) \rightarrow 1$. The contact resistance is also plausible from the perspective that the voltage sources, which influence only the electrochemical potential deep inside the reservoirs, have no way of knowing that an electron has, in fact, transited the device until it comes into equilibrium with a Fermi function inside the reservoir. While the above argument is only heuristic, we believe it can be placed on firmer mathematical grounds by analogy with the onset of resistance in narrow superconductors via "phase slips."

The factor of $T(E)$ which enters into Eq. (8) can also be understood from an energy dissipation viewpoint: Only those electrons which in fact transit the device dissipate energy and wind up contributing to the resistance. We also note that the analog of Eq. (9) at finite voltages has been suggested in Ref. 11 to be $I/V_d = (2e^2/\hbar)(\bar{T}/\bar{R})$, where \bar{T} is simply the energy averaged transmission coefficient defined in Eq. (7) and $\bar{T} + \bar{R} = 1$.

Consider now a quasi-one-dimensional ballistic wire described by the separable potential $U(x, y, V) = U_x(x, V) + U_y(y)$, where $U_y(y)$ is a confining potential which quantizes the electron energies perpendicular to the direction of transport. The separable Hamiltonian implies that there is no scattering between the different subbands created by the confining potential $U_y(y)$. Equation (7) therefore generalizes, as we show in Sec. IV, to simply a sum over all the possible paths for electrons to transmit between the left and right contacts:

$$I(E, V, T) = \frac{e}{\pi\hbar} \sum_i T_i(E, V) \otimes W(E, V) \otimes \left[-\frac{df}{dE}(E, T) \right], \quad (13)$$

where the sum is over all occupied subbands. For a ballistic quasi-one-dimensional wire with $U_x(x, V) \simeq 0$, we have

$$T_i(E, V) \simeq \theta(E - E_i) \quad (14)$$

at moderate voltages. Here E_i is the subband energy for quantized motion perpendicular to the direction of transport. The limiting case of this formula at zero temperature is when the thermal smearing function becomes a δ function so that

$$I(E, V, T=0) = \frac{e}{\pi\hbar} \sum_i \theta(E - E_i) \otimes W(E, V). \quad (15)$$

We can carry out the convolution to obtain

$$\theta(E - E_i) \otimes W(E, V) = \begin{cases} 0, & E < E_i - eV \\ E - (E_i - eV), & E_i - eV < E < E_i \\ eV, & E > E_i \end{cases} \quad (16)$$

We sketch the current in a ballistic quasi-one-

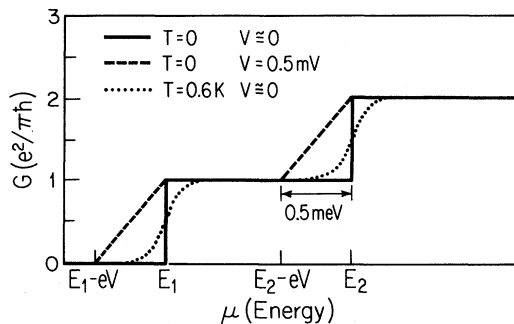


FIG. 3. Ballistic contact resistance in a quasi-one-dimensional quantum wire illustrating the effects of finite voltages and finite temperature. The conductance is shown versus the electrochemical potential μ in the right contact. The finite width of the steps in conductance is inversely proportional to the phase coherence time introduced by finite voltage or finite temperature.

dimensional wire with several subbands versus the electrochemical potential μ in the right contact in Fig. 3. The conductance G is defined as the current divided by voltage, even at finite voltages. These steps in the ballistic conductance have been observed^{21,22} and explained using formalisms similar to the one in this paper as quantum contact resistances. The conductance at finite voltages, given by the dashed line in Fig. 3, shows initially a linear dependence versus Fermi energy instead of a step at zero temperature. We also show the effect of finite temperature on the conductance in Fig. 3 for comparison. Both finite temperature and finite voltage smear out the sharp steps in the zero-temperature conductance. The maximum slope of these steps in the conductance versus Fermi energy occurs at conductance values half-way between the plateaus, and is given by

$$\left. \frac{dG}{d\mu} \right|_{\max} = \frac{e^2}{\pi\hbar} \frac{1}{4k_B T} \approx \frac{e^2}{\pi\hbar} \frac{\tau_T}{\hbar} \quad \text{for } V \approx 0 \quad (17)$$

at finite temperature and

$$\left. \frac{dG}{d\mu} \right|_{\max} = \frac{e^2}{\pi\hbar} \frac{1}{eV} \approx \frac{e^2}{\pi\hbar} \frac{\tau_V}{\hbar} \quad \text{for } T=0 \quad (18)$$

at finite voltages. The maximum slope of the steps in conductance versus Fermi energy is limited by the phase coherence times τ_T and τ_V , and is proportional to the phase coherence time. In a field effect transistor where the chemical potential is proportional to the gate bias voltage V_g , and if the chemical potential were controlled by such a gate, one could interpret $g_m = dI/dV_g = eV dG/d\mu$ as the transconductance of the device. For such a structure, the transconductance would have a finite maximum value even in the ballistic transport regime. When the applied voltage is much greater than the temperature, this maximum transconductance in the quasi-one-dimensional ballistic wire becomes just $g_m = 2e^2/h$.

III. CLASSICAL DIFFUSIVE LIMIT OF LANDAUER'S FORMULA

We are interested in the case with there are n scatterers randomly distributed in one dimension between two reservoirs as in Fig. 4. We consider the diffusion of classical particles down the chain of scatterers. Evaluating Landauer's formula for this case requires evaluating the transmission probability of a classical particle down a chain of n identical scatterers. This can be treated as a classical random walk down the n scattering sites. The classical transmission probability T_n down the total chain of n scatterers can be obtained in terms of a recursion relation

$$T_n = \frac{T_1 T_{n-1}}{1 - R_1 R_{n-1}} \quad \text{where } T_n + R_n = 1. \quad (19)$$

The solution of this recursion relation, which can be motivated from physical arguments involving the addition of classical resistors in series,² is

$$T_n = \frac{1}{(R_1/T_1)n + 1}, \quad (20)$$

where T_1 is the probability of a classical particle being transmitted by a single scatterer. This result has been obtained in Ref. 2, and the special case with $T=R=\frac{1}{2}$ treated in Ref. 23 using a different method. Substituting this classical transmission coefficient into Landauer's conductance formula in one dimension, Eq. (8), gives the diffusive conductance

$$G_{1D} = \frac{e^2}{\pi\hbar} \frac{1}{(R_1/T_1)n + 1}. \quad (21)$$

If we chose the scattering centers to be different, with scatterer a having a transmission coefficient T_a , etc., we obtain for the resistance R_{1D} after solving a recursion relation similar to Eq. (19)

$$\frac{e^2}{\pi\hbar} R_{1D} = \frac{R_a}{T_a} + \frac{R_b}{T_b} + \frac{R_c}{T_c} + \dots + 1, \quad (22)$$

which is just the classical sum of the original R/T Landauer resistance from each scatterer plus the contact resistance.

If we choose $T_a = R_a = \frac{1}{2}$ for each scatterer, corresponding to the assumption of isotropic scattering in the

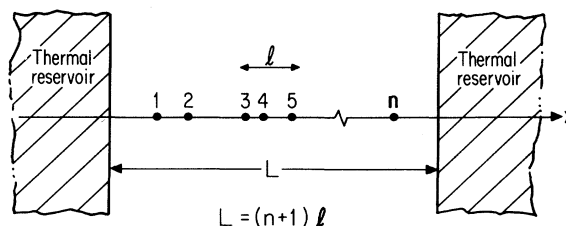


FIG. 4. n randomly positioned scatterers between two thermal reservoirs in one dimension. The two reservoirs are also regarded as scatterers so that the mean separation l between scattering centers is $L = (n+1)l$, where L is the sample length.

conductor, each elastic collision with a scatterer completely randomizes the particle's momentum direction. Our form for the classical diffusive conductance, Eq. (22), suggests that we interpret the contact resistance on the right-hand side as stating that the thermal reservoirs completely randomize the momentum direction of the electron when it enters the reservoir. The reservoir captures all particles incident upon it, then immediately randomizes their momentum direction due to rapid scattering inside the reservoir. This assumption is included in our derivation of the conductance formula (8) by assuming the reservoirs are described by a Fermi distribution.

We now wish to compare Eq. (21) to the standard Drude-Sommerfeld conductivity expression in one dimension,

$$\sigma_{1D} = e^2 N(E_F) D(E_F) = e^2 \frac{2}{\pi \hbar} v_F \tau_{tr} = 2 \frac{e^2}{\pi \hbar} l_{tr}, \quad (23)$$

where $l_{tr} = v_F \tau_{tr}$ is the transport mean free path at the Fermi level and $D(E_F) = v_F^2 \tau_{tr}$ is the diffusion coefficient. To obtain the conductance in one dimension, we merely divide by the length of the sample:

$$G_{1D} = \frac{\sigma_{1D}}{L} = 2 \frac{e^2}{\pi \hbar} \frac{l_{tr}}{L}. \quad (24)$$

In order for the diffusive conductance from Landauer's formula, Eq. (21), to equal the Drude conductance in Eq. (24), we must conclude that

$$2 \frac{l_{tr}}{L} = \frac{1}{(R_1/T_1)n + 1}. \quad (25)$$

To show that Eq. (25) is correct, consider the special case of isotropic scattering in the conductor. Here $T_1 = R_1 = \frac{1}{2}$, so that the transport mean free path is equal to the mean distance between the scatterers, i.e., $l_{tr} = l$. In that case Eq. (25) becomes

$$2 \frac{l}{L} = \frac{1}{n + 1}. \quad (26)$$

In the limit that n becomes large, Eq. (26) implies that the mean free path is half the average spacing between scattering centers. We would obtain this result for the mean free path in one dimension if we average the distance to collide with a scatterer over a uniform distribution of all possible starting positions for the random walk. This statistical average is consistent with the standard assumption used in deriving the Drude conductivity, in which we consider an electron executing a random walk inside an "average" conductor that is spatially uniform. We can either keep the scatterers fixed and average over all possible starting positions for the random walk, or we can keep the origin of the random walk fixed and average over all possible impurity locations. Either way we obtain that the mean free path which should appear in the Drude formula, Eq. (24), is half the average spacing between scatterers. Consequently, for large n , Eq. (26) is correct, so that the Drude conductance is equivalent to the diffusive limit of the Landauer conductance. Moreover, even for small n , Eq. (26) can be shown to be valid by the following argument: Consider again

the arrangement of the n scatterers between two thermal reservoirs as shown in Fig. 4. The reservoirs are regarded as being two additional scatterers which absorb all carriers incident upon them while simultaneously randomizing the carrier's momentum direction, so they must be included in computing the mean free path l . From the geometry of the scatterers we therefore have for a given scattering configuration that $L = (n + 1)l$. The mean free path in the average configuration of the scatterers is therefore $L = 2(n + 1)l$ as in Eq. (26). We see again that the Drude conductance, given by Eq. (24), and Landauer's conductance formula, Eq. (8), in the classical diffusive limit are the same.

The Drude conductance formula and the Landauer conductance formula can also be made equivalent in the ballistic limit of no scattering between the thermal reservoirs ($n = 0$), if we reinterpret the meaning of the transport mean free path which enters the Drude conductance in Eq. (24). The standard interpretation of the transport mean free path $l_{tr} = v_F \tau_{tr}$ is that τ_{tr} is the scattering time in an infinite system. In this standard interpretation l_{tr} can increase without bound. If we now reinterpret l_{tr} as the average distance for the conduction electron to randomize its momentum direction in a finite sample, we see that the longest possible transport mean free path for any given sample is just the sample length. For the average sample described by Eq. (24), this argument gives $2l_{tr} = 2l = L$, in which case the ballistic conductance is

$$G_{1D} = \frac{e^2}{\pi \hbar} \quad (27)$$

as before. The contact resistance can be thought of from this viewpoint as a consequence of the electron's transport mean free path being limited to at most the sample length.

We can gain further insight into the Drude-Sommerfeld conductance by considering a quantum particle moving in a set of potential barriers in one dimension, with barrier a having a quantum-mechanical transmission probability T_a , etc. In computing the total transmission probability of a quantum particle down this chain of scatterers one cannot ignore the phase of the particle. The total transmission coefficient down the chain will therefore be much more complicated than in Eq. (22). Büttiker⁵ has worked out the general case for the addition of quantum resistors in series. In the case where the particle inelastically scatters with probability one between each elastic scatterer, his results can be written very simply as

$$\frac{e^2}{\pi \hbar} R_{1D} = \left[\frac{R_a}{T_a} + 1 \right] + \left[\frac{R_b}{T_b} + 1 \right] + \left[\frac{R_c}{T_c} + 1 \right] + \cdots + 1. \quad (28)$$

The quantum particle picks up the contact resistance each time it inelastically scatters. Equation (28) was derived for zero applied voltage. In the case of a finite voltage, we must average each transmission and reflection coefficient over the energy dissipated into the inelastic scatterer. This energy averaging will be of the order of

the applied voltage divided by the number of additional inelastic scatterings inside the sample. If the number of inelastic scattering events increases with temperature, then the energy averaging due to voltage and temperature will be related, in contrast to the case where there is no inelastic scattering inside the device. Another interesting property of Eq. (28) is that the total transmission coefficient decreases linearly with the length of the 1D chain as expected if the motion is diffusive. It is generally believed that, for most types of disorder, the resistance of a 1D chain without inelastic scattering increases exponentially with the length of the chain.²⁴ Therefore, it appears that inelastic scattering processes are necessary to obtain diffusion in 1D. Equation (28), if compared with the Drude conductance in Eq. (24), gives Matthiessen's rule $l_{\text{tr}}^{-1} = l_{\text{el}}^{-1} + l_{\text{in}}^{-1}$, where $L/2l_{\text{el}} = (R_1/T_1)n$ for the case of identical barriers and $L/2l_{\text{in}} = n + 1$.

IV. TUNNELING LIMIT OF LANDAUER'S FORMULA IN TWO AND THREE DIMENSIONS

We consider here the generalization of Landauer's conductance formula to include many quantum channels or sets of good quantum numbers in the ideal device leads as described in Refs. 3 and 10. Define the current transmission probability T_{ij} from channel j in the left reservoir to channel i in the right reservoir evaluated at a given energy as

$$T_{ij} = \frac{J_{i,\text{right}}}{J_{j,\text{left}}} = t_{ij}t_{ij}^* \quad (29)$$

when all other incoming currents are zero. Here t_{ij} is the current transmission amplitude as defined in the scattering matrix.¹⁰ The current from left to right, in the spirit of the derivation in Ref. 14, is therefore obtained as

$$I_{\text{lr}} = \sum_{i,j} e \int v_j^+(E) N_j^+(E) T_{ij}(E, V) \times f_1(E - (\mu + eV)) [1 - f_r(E - \mu)] dE. \quad (30)$$

The current from right to left is similarly

$$I_{\text{rl}} = \sum_{i,j} e \int v_i^-(E) N_i^-(E) T_{ji}(E, V) f_r(E - \mu) \times [1 - f_1(E - (\mu + eV))] dE. \quad (31)$$

Here $v^+ = v^-$ and $N^+ = N^- = N/2$.

Current conservation along with time reversal symmetry give the constraint $t_{ij} = t_{ji}$, implying that $T_{ij} = T_{ji}$. The net current is therefore

$$I = I_{\text{lr}} - I_{\text{rl}} = \sum_{i,j} \frac{e}{\pi\hbar} \int T_{ij}(E, V) [f_1(E - (\mu + eV)) - f_r(E - \mu)] dE. \quad (32)$$

Equation (32) reduces to the general result

$$I = \frac{e}{\pi\hbar} \sum_{i,j} T_{ij}(E, V) \otimes W(E, V) \otimes \left[-\frac{df(E, T)}{dE} \right], \quad (33)$$

where $W(E, V)$ is again the voltage window given by

$$W(E, V) = [\theta(E + eV) - \theta(E)].$$

Equation (33) shows that temperature broadening and voltage broadening are statistically independent even in this multichannel case. Equation (33) reduces at low temperatures and small voltages to the well-known result^{3,4,8}

$$I = \frac{e^2 V}{2\pi\hbar} \sum_{i,j} T_{ij} = \frac{e^2 V}{2\pi\hbar} \sum_{i,j} t_{ij}t_{ij}^* = \frac{e^2 V}{2\pi\hbar} \text{Tr}(tt^\dagger), \quad (34)$$

where the sum in Eq. (34) also runs over the individual spin quantum numbers, thus differing by a factor of 2 from Eq. (33). Equation (34) is analogous to the Landauer formula proportional to T , Eq. (8) describing a two-probe measurement. References 3, 6, and 10 give the multiple channel Landauer formula which describes a four-probe measurement, analogous to the T/R Landauer formula in Eq. (9).

For a general T_{ij} , Eq. (34) allows the possibility that electrons can scatter between different quantum channels. We are interested in the tunneling limit of this formula, for which $T_{ij} = T_i \delta_{ij}$. The current transmission probability takes this form only if the Hamiltonian is separable as $\hat{H}_{\text{total}} = \hat{H}_x(x) + \hat{H}_y(y) + \hat{H}_z(z)$, thus justifying Eq. (13) in our earlier treatment of the ballistic quasi-one-dimensional wire. This tunneling limit corresponds to the classical addition of resistors in parallel, i.e., there is no communication between the parallel resistors. Because there is no scattering between the different conducting channels inside the sample, we only have to sum over one channel index in Eq. (34), namely,

$$I = \frac{e^2 V}{2\pi\hbar} \sum_{i,j} T_i \delta_{ij} = \frac{e^2 V}{2\pi\hbar} \sum_i T_i. \quad (35)$$

This limit also corresponds to the usual treatment of electron tunneling, for which we assume a static potential having spatial variation only along the x direction.^{12,14,15} To prove this, note first that the Hamiltonian is separable so that $E = E_x + E_y + E_z$. Furthermore, the eigenfunctions along y and z are plane waves so that $E_y = \hbar^2 k_y^2 / 2m$ and $E_z = \hbar^2 k_z^2 / 2m$. The transverse channel index i can be labeled by $i = (k_y, k_z)$. Translational invariance along the y and z directions implies there is no scattering potential to couple the transverse channels. We enforce conservation of energy by writing

$$T_i(E_x) = T(E - E_y(k_y) - E_z(k_z)).$$

Expressing the multichannel Landauer formula, Eq. (35), in terms of a current density then gives

$$J_{3D} = \frac{I}{L_y L_z} = \frac{e^2 V}{2\pi\hbar} \frac{1}{L_y L_z} \sum_{k_y, k_z, \text{spin}} T(E - E_y(k_y) - E_z(k_z)). \quad (36)$$

Recognizing the one-dimensional identity

$$2\frac{1}{L_y} \sum_{k_y} \rightarrow \int N_{1D}^{\text{free}}(E_y) dE_y, \quad (37)$$

where $N_{1D}^{\text{free}}(E)$ is the one-dimensional free-electron density of states including spin

$$N_{1D}^{\text{free}}(E) = \frac{1}{\pi} \left[\frac{2m}{\hbar^2} \right]^{1/2} \left[\frac{1}{E} \right]^{1/2} \theta(E), \quad (38)$$

the current density now becomes

$$J_{3D} = \frac{e^2 V}{2\pi\hbar} \sum_{\text{spin}} \int \int \frac{1}{2} N_{1D}^{\text{free}}(E_y) \frac{1}{2} N_{1D}^{\text{free}}(E_z) \\ \times T(E - E_y(k_y) - E_z(k_z)) dE_y dE_z. \quad (39)$$

Equation (39) can be written as a convolution

$$J_{3D} = \frac{I}{L_y L_z} = \frac{e^2 V}{\pi\hbar} T(E) \otimes \frac{1}{2} N_{1D}^{\text{free}}(E) \otimes \frac{1}{2} N_{1D}^{\text{free}}(E). \quad (40)$$

If we had started from a standard tunneling formalism,^{12,14} we would have obtained

$$J_{3D}(E, V, T) = \frac{e}{\pi\hbar} T(E, V) \otimes W(E, V) \otimes \left[-\frac{df}{dE}(E, T) \right] \\ \otimes \frac{1}{2} N_{1D}^{\text{free}}(E) \otimes \frac{1}{2} N_{1D}^{\text{free}}(E) \quad (41)$$

as the current density at finite voltage and finite temperature. Equation (41) is shown graphically in Fig. 2. From Eq. (41) one can obtain any tunneling current at finite voltage and temperature found in Refs. 12, 14, and 15. A helpful identity in obtaining these tunneling currents is

$$\frac{1}{2} N_{1D}^{\text{free}}(E) \otimes \frac{1}{2} N_{1D}^{\text{free}}(E) = \frac{1}{2} N_{2D}^{\text{free}}(E) = \frac{1}{2} \frac{m}{2\pi\hbar^2} \theta(E). \quad (42)$$

Results analogous to Eqs. (42) and (40) for the density of states and electrical currents follow for any separable potential.¹³ Using Eqs. (42) and (40), we see that in the standard type of quantum-mechanical tunneling calculations in three dimensions, the quantum contact resistance never appears. The reason is, of course, that in going over to a continuum of states we add an infinite number of conducting channels in parallel. For tunneling at finite voltages, Eq. (33) reduces to the standard tunneling current density at finite voltages, Eq. (41), thus having the correct behavior in the tunneling limit.

By analogy with our consideration of classical diffusion

in the previous section, we note that the multiple-channel Landauer formula has been applied to consider voltage fluctuations in small diffusive conductors.²⁵

V. CONCLUSIONS

Using the Landauer formalism we have examined the effects of finite voltages on quantum transport, showing that the finite voltage acts to average the electrical conductance over an energy range equal to the applied voltage. This voltage broadening is similar to ordinary thermal broadening of the conductance, which averages the conductance over an energy range equal to the temperature. We showed that both thermal and voltage broadening of the conductance can be expressed as independent convolutions with known functions, proving that voltage broadening and ordinary temperature broadening are statistically independent. Motivated by the recent experiments of van Wees *et al.*²¹ and Wharam *et al.*,²² we examined thermal and voltage broadening of the conductance using a quasi-one-dimensional ballistic conductor as a model system. The finite width of the rises in the quantized steps in conductance versus electrochemical potential in these experiments is inversely proportional to the phase coherence time of the electron introduced by either finite voltage or finite temperature. The quantum contact resistances can be thought of either as a consequence of the rate limitation imposed by the uncertainty principle in the device leads or as a consequence that the mean free path is limited at most to the device size.

We also examined both the diffusive and tunneling limits of Landauer's conductance formula. In the classical diffusive limit the one-dimensional Landauer formula reduces to the Drude-Sommerfeld conductance in one dimension. In the tunneling limit, the finite-voltage Landauer formula is equivalent to many previous treatments of electron tunneling at finite voltages. In this tunneling limit, we have shown that simply by repeatedly convolving the tunneling current in one dimension with the one-dimensional free electron density of states, one obtains the tunneling current density in two and three spatial dimensions.

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$$W(E, V) = \begin{cases} [\theta(E + eV) - \theta(E)], & eV < \mu \\ [\theta(E + \mu) - \theta(E)], & eV \geq \mu. \end{cases} \quad (43a)$$

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tance of one electron of each spin in a device of length L_d by

$$I = \frac{2e}{\tau_d} = e^2 D(E_F) N(E_F) \frac{V}{L_d}, \quad (44)$$

where V is the applied voltage and $D(E_F) = v_F^2 \tau_F$ is determined by the Fermi velocity and momentum relaxation time. We later show that Eq. (44) is equivalent to Eq. (9) in the classical diffusive limit. The time τ_d is then given as

$$\frac{2}{\tau_d} = D \frac{N(E_F) eV}{L_d} = \left[\frac{D}{L_d^2} \right] [N(E_F) eV L_d] \quad (45)$$

describing an electron gas in one dimension of arbitrary density $n = N(E_F) eV$. Thus τ_d is the time for one electron to diffuse across the device divided by the number of electrons in the device.

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