Excitation Errors in Phased Arrays

Manufacturing tolerances, unit-to-unit variation of electronic components, thermal/shot noise, etc., cause array excitations to differ from the desired values. These variations affect the radiation pattern (shape, gain, side lobe level).
Excitation errors result in gain loss (typically < 1 dB) and higher sidelobe levels (may increase 10 dB or more!).

Consider a linear array (for simplicity). (This derivation is similar to Elliott, IEEE Trans. Ant. Prop., January 1958.)

\[ E(\theta) = K \frac{e^{-j\phi_0 \theta}}{\theta} \sum A_n e^{j\phi_n} \]

\[ \phi = \phi_0 d (\sin \theta - \sin \theta_0) \]

Note that \( A_n \) is real because the phase (progressive in the example) is \( -\theta d \sin \theta_0 \).

\[ A_n = A_n (1 + \xi_n) e^{j \phi_n} \]

Real, small, independent, Gaussian random variables with zero mean & variances \( \sigma_\xi^2 = \frac{\Delta^2}{\delta^2} \) and \( \sigma_\phi^2 = \frac{S^2}{\delta^2} \).
For $\phi_n$ small, $e^{i\phi_n} \approx 1 + j\phi_n$, so

$$\tilde{A}_n \approx A_n \left(1 + \delta_n \right) \left(1 + j\phi_n\right) \approx A_n \left(1 + \delta_n + j\phi_n\right)$$

Then,

$$\tilde{E}(\rho) \approx \tilde{E}_0 \left\{ \sum_n A_n e^{in\psi} + \sum_n \phi_n A_n e^{in\psi} \right\} + \frac{k e^{jkr}}{r^2} f_c(\theta\phi) + j \sum_n \phi_n A_n e^{in\psi}$$

Look at the power pattern, i.e., $|\tilde{E}|^2$

$$|\tilde{E}|^2 = \tilde{E} \tilde{E}^* = |\tilde{E}_0|^2 \left\{ \sum \frac{A_n A_n^*}{n} \right\} + \sum \frac{A_n A_n^*}{n} \sum \frac{A_m A_m^*}{m} + \sum \frac{A_n A_n^*}{n} \sum \frac{A_n A_n^*}{m} + \sum \frac{A_n A_n^*}{m} \sum \frac{A_n A_n^*}{m}$$

$$= |\tilde{E}_0|^2 \left\{ \sum \sum_n A_m A_n e^{i(n-m)\psi} \right\}$$

$$+ j \sum \sum_n \phi_n A_m A_n \left[ e^{i(n-m)\psi} - e^{-i(n-m)\psi} \right]$$

$$+ \sum \sum_n \delta_m A_m A_n \left[ e^{i(n-m)\psi} + e^{-i(n-m)\psi} \right]$$

$$+ \sum \sum_n \phi_m \phi_n A_m A_n e^{i(n-m)\psi}$$

$$+ j \sum \sum_n \phi_n \delta_m A_m A_n \left[ e^{i(n-m)\psi} - e^{-i(n-m)\psi} \right]$$

$$+ \sum \sum_n \delta_m \delta_n A_m A_n e^{i(n-m)\psi}$$
We will look at the expected value of this random variable $|E|^2$.

Recall that the $\delta_n$ and $\phi_n$ are independent and have zero mean. Therefore,

$$E[|E|^2] = |E_0|^2 \sum_{m,n} A_m A_n e^{j(m-n)\psi}$$

$$+ |E_0|^2 \sum_m \sigma_m^2 A_m^2 + \sum_m \sigma_m^2 A_m^2$$

$$= |E|^2 + |E_0|^2 (\bar{\phi}^2 + \Delta^2) \sum_m A_m^2$$

Now, let's look at Sidelobe Levels!

The exact design yields,

$$S = \frac{|E(\phi, \phi)|^2}{|E_{\text{max}}|^2} = \frac{\sum_{m,n} A_m A_n e^{j(m-n)\psi}}{(\sum_m A_m)^2}$$
The perturbed design yields,

$$
S = \frac{\sum_n \sum_m A_m A_n e^{j(n-m)\psi}}{(\sum_m A_m)^2} + (\bar{\phi}^2 + \bar{\Delta}^2) \frac{\sum_m A_m^2}{(\sum_m A_m)^2} \\
$$
\[\text{Assume beam max is not changed}\]

$$
\tilde{S} = S + (\bar{\phi}^2 + \bar{\Delta}^2) \frac{\sum_m A_m^2}{(\sum_m A_m)^2}
$$

\[\tilde{S} > S \text{ if } \bar{\phi}^2 \text{ or } \bar{\Delta}^2 > 0. \text{ The amount of sidelobe increase is}\]

$$
R = \frac{\tilde{S}}{S} = 1 + \frac{\bar{\phi}^2 + \bar{\Delta}^2}{S} \frac{\sum_m A_m^2}{(\sum_m A_m)^2}
$$

Usually, we express SLL in dB,

$$
\delta = 10 \log_{10} S \Leftrightarrow S = 10^{\delta/10}
$$

$$
R = 10 \log_{10} \left\{ 1 + 10^{-\delta/10} (\bar{\phi}^2 + \bar{\Delta}^2) \frac{\sum_m A_m^2}{(\sum_m A_m)^2} \right\}
$$
Note that
\[ \frac{\sum_m A_m^2}{(\sum_m A_m)^2} = \frac{1}{D} \quad , \quad d = \frac{\lambda_0}{2a} \]

Therefore,
\[ R = 10 \log_{10} \left[ 1 + 10^{-\frac{\gamma_0}{10}} \left( \frac{\Phi^2 + \Delta^2}{D} \right)^2 \right] \quad \text{linear array} \]

The comparable expression for planar arrays is
\[ R = 10 \log_{10} \left[ 1 + 10^{-\frac{\gamma_0}{10}} \left( \frac{\Phi^2 + \Delta^2}{D} \right)^2 \right] \quad \text{planar array} \]
Now, consider an array for which the errors dominate the sidelobe performance.

Recall,

$$\mathcal{E} |\vec{E}|^2 \approx |\vec{E}_0|^2 \sum_m (\sigma^2 + \Delta^2) A_m^2$$

↑ Very small
   in SL region

Therefore, sidelobes are comprised of the sum of the independent error signals radiating from each element.

$$\mathcal{E} |\vec{E}|^2 \approx |\vec{E}_0|^2 \sum_m \sigma^2 A_m^2 + |\vec{E}_0|^2 \sum_m \Delta^2 A_m^2$$

Recall the expression for sidelobe level,

$$\hat{S} = S + (\sigma^2 + \Delta^2) \frac{\sum A_m^2}{(\sum A_m)^2}$$

↑ Small by design dominates
Therefore,

\[ \tilde{S} \sim (\Phi^2 + \Delta^2) \frac{\sum_m A_m^2}{(\sum_m A_m)^2} \]

\[ = (\Phi^2 + \Delta^2) \frac{\tilde{S}}{D^2} \]

, linear array

\[ = (\Phi^2 + \Delta^2) \left( \frac{\tilde{S}}{D} \right) \]

, planar array

Remember that our definition of \( \tilde{S} \) makes it relative to peak gain of the array, i.e., relative to \( D \).

We can find expression for the average isotropic sidelobe level from \( \tilde{S} \) if we multiply by \( D \).

\[ \overline{SLL}_{dB} = 10 \log_{10} \left[ \Phi (\Phi^2 + \Delta^2) \right] \]

planar array where errors dominate SLL
Figure 15. Typical Pattern of Ultralow Sidelobe Dipole Array

Comments & Notes:

1. $\phi_{\text{rms}} = \sqrt{\langle \phi^2 \rangle}$ is in radians and $\Delta_{\text{rms}} = \sqrt{\langle \Delta^2 \rangle}$ is w.r.t. unity.

2. All of the expressions for SLL are expected values of a random quantity, i.e., average sidelobe level. Often, the peak SLL is important. For large arrays with small random (uncorrelated) errors, $\text{SLL}_{\text{peak}} \approx \overline{\text{SLL}} + 5\text{ dB}$ is a reasonable estimate. There is more discussion of peak SLL in terms of probability density function in the text.
3. Consider the errors allowed to achieve \(-45\) dB relative average SLL for a planar array with 30 dBi gain.

\[
SLL_{dBi} = 30 - 45 = -15\text{ dBi}
\]

\[
= 10 \log_{10} \left[ \frac{1}{10} \left( \phi^2 + \Delta^2 \right) \right]
\]

\[
\frac{\Delta^2}{\phi^2 + \Delta^2} = 10^{-15/10} = 0.0316
\]

\[
\Rightarrow \phi^2 + \Delta^2 = 0.01
\]

No amplitude error \(\Rightarrow \phi_{rms} = 0.1 \approx 6^\circ\)

No phase error \(\Rightarrow \Delta_{rms} = 0.1 \approx 0.9\text{ dB}\)

\[
10 \log_{10} (1-\Delta_{rms})^2
\]

The relationship of \(\phi^2\) and \(\Delta^2\) to \(SLL\) (average isotropic SLL) has a handy graphical interpretation.

\[
\phi^2 + \Delta^2 = \frac{1}{10} 10^{-45/10} + \text{circle}
\]
Ruze presented a chart like the one below, Fig 7.1 of text.

Figure 7.1  Array average (residual) sidelobes (relative to isotropic radiation) due to amplitude error. Element gain $\pi$ assumed. (After: [11].)
4. It is easier to obtain low relative sidelobes with a large, high-gain array than with a low-gain array.

Suppose the example above is modified: Obtain $-45 \text{ dB}$ average relative SLL, but use array with $40 \text{ dB}$ gain

$$\text{SLL}_\text{dB} = 40 - 45 = -5 \text{ dB}$$

$$= 10 \log_{10} [\frac{1}{\sqrt{(\theta^2 + \Delta^2)}}]$$

$$\therefore \theta^2 + \Delta^2 \approx 0.316$$

No amplitude error $\Rightarrow \phi_{\text{rms}} \approx 18^\circ$

(3x previous example)

Based on these considerations and based on achievable error in typical microwave components, arrays are sometimes classified as below.
5. The two approaches to error effects agree for arrays where SLL is dominated by errors.

\[ \text{SLL}_\mathrm{dB} = 10 \log_{10} \left( \frac{\pi (\varphi^2 + \Delta^2)}{\tau} \right) \]

\[ \varphi = 10 \log_{10} \left( 1 + 10^{-\frac{d}{10}} \frac{\tau (\varphi^2 + \Delta^2)}{D^3} \right) \]

Suppose \( \varphi^2 + \Delta^2 = 0.01 \), \( D = 30 \text{dBi} \) and \( \varphi = -55 \text{dB} \).
\[
\text{SLL}_\text{dB}_i = 10 \log_{10} \left[ \pi \left( 0.01 \right) \right] = -15 \text{ dB}_i \\
R = 10 \log_{10} \left[ 1 + 10^{+5.5} \frac{\pi \left( 0.01 \right)}{10^3} \right]^\frac{3}{2} \\
= 10 \log_{10} \left[ 1 + 9.93 \right] = 10.4 \text{ d B} \\
\text{Design SLL} = -5.5 \text{ d B} \\
\text{Achieved SLL} = -44.6 \text{ d B} \\
\text{Achieved SLL}_{\text{dB}_i} = 30 \text{ d B}_i - 44.6 \text{ d B} \\
= -14.6 \text{ d B}_i
\]

6. Recall that the error signals radiating from the elements are assumed to be independent, so those radiated powers spread somewhat evenly (according to element pattern) throughout space. This "uniform" power level sets minimum achievable SLL for all angles.

If the errors are correlated over some distance, the distribution of SLL power is affected.
Figure 10. Effects of Various Errors

Effects of Phase Errors on Array Gain

In the developments for SLL, we stated that gain loss is usually small and we used \( |\tilde{E}_{\text{max}}| \approx |\tilde{E}_{\text{max}}| \) for one of the derivations.

Consider an array with

\[
\tilde{A}_m = A_m e^{j\phi_m} \quad \text{independent, Gaussian, zero mean}
\]

\[
\tilde{E} = \tilde{E}_0 \sum_n A_n e^{j\phi_n} e^{jn\psi}
\]

The main beam occurs when \( \psi = 0 \), so

\[
\tilde{E}_{\text{max}} = \tilde{E}_0 \sum_n A_n e^{j\phi_n}
\]

For simplicity, consider uniform illumination, \( A_n = A_0 \).

\[
\tilde{E}_{\text{max}} = A_0 \tilde{E}_0 \sum e^{j\phi_n} = A_0 \tilde{E}_0 \sum \cos \phi_n + j \sum \sin \phi_n
\]
Since $\phi_n$ has zero mean, the second $\Sigma$ goes to zero. For small $\phi_n$,

$$\bar{E}_{\text{max}} \approx A_0 \bar{E}_0 \sum_{n=1}^{N} (1 - \frac{\phi_n^2}{2})$$

$$\varepsilon \xi |\bar{E}_{\text{max}}|^2 \approx NA_0^2 \bar{E}_0^2 (1 - \phi_{rms}^2 + \frac{\phi_{rms}^4}{4})$$

Ignore this term.

For fixed radiated power, $G \propto |\bar{E}_{\text{max}}|^2$,

so

$$\frac{\Delta G}{G} \approx -\phi_{rms}^2$$

Example: Suppose $\phi_{rms}^2 = 0.2 \approx 11.5^\circ$

$$\frac{\Delta G}{G} = -\phi_{rms}^2 = -0.04$$

$$\frac{G}{G} = \frac{G + \Delta G}{G} = 1 + \frac{\Delta G}{G} = 0.96 = -0.18 \text{ dB}$$

This phase error with no amplitude error causes SLL.
\[ \text{SIL dB} = 10 \log_{10} \left[ \pi \left( 0.2 \right)^2 \right] = -9 \text{dB} \]

That is, 11.5° rms phase error reduces the main beam gain by ≈ 0.2 dB and redistributes the power throughout space to create a floor for SIL at approx. -9 dB}.