An alternative formulation for planar infinite array. This formulation is particularly useful for MOM analysis.

Suppose we have a single infinitesimal dipole at \((x_0, y_0)\) in the \(z=0\) plane.

\[
\mathbf{J}(\mathbf{F}) = \iota \delta (x-x_0) \delta (y-y_0) \delta (z)
\]

\[
\mathbf{A}(\mathbf{F}) = \frac{1}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{F}') e^{-j \mathbf{r} \cdot \mathbf{r}'}}{\mathbf{r} - \mathbf{r}'} \, d\mathbf{v}'
\]

\[
\mathbf{A}_{00} = \frac{x}{4\pi} \frac{e^{-j \sqrt{\omega \mu} \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}}}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}}
\]
The electric field is

\[ \overline{E} = \frac{1}{j \omega \mu} \left( \nabla \times \nabla \times \mathbf{A} - \mathbf{J} \right) \]  (3)

Now, suppose we have an infinite array of these infinitesimal dipoles with progressive phase so the current in \((m,n)\) cell is

\[ J_{mn} = J_0 e^{-j \kappa d x_m} e^{-j \kappa d y_n} \]  (4)
The vector potential due to this infinite array of infinitesimal dipoles is

\[
\mathbf{A}_{\text{i.d.}}(x, y, z) = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j2\mu_0 m d x} e^{-j2\nu_0 n d y} \frac{e^{-j\omega t \sqrt{(x-x_0-md_x)^2 + (y-y_0-md_y)^2 + z^2}}}{\sqrt{(x-x_0-md_x)^2 + (y-y_0-md_y)^2 + z^2}}
\]  

(5)

Exercise: Show

\[
\mathbf{A}_{\text{i.d.}}(x+pd_x, y+qdy, z) = \mathbf{A}_{\text{i.d.}}(x, y, z) e^{-j2\mu_0 p d x} e^{j2\nu_0 q d y}
\]

Any B.C. satisfied in \((0, 0)\) unit cell is satisfied in all \((p, q)\) cells, with a progressive phase shift.

\[
\mathbf{A}_{\text{i.d.}}(x, y, z) \text{ in (5) is the Green's function that can be used to find the fields everywhere due to a specified current distribution on conductors in the } Z=0 \text{ plane.}
\]
For example, suppose

$$\mathbf{J} = x J_{oo}(x,y) \quad \text{in} \ (0,0) \ \text{cell} \ (6)$$

and satisfies (4) in all cells. ($J_{oo}(x,y)$ might be of the form $\cos \frac{\pi x}{L}$ on a string dipole of length $L$.)

Then, the vector potential for the infinite array is

$$A_{\text{array}}(x,y,z) = \frac{x}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi}{L} (mx + ny)} e^{-j q z}$$

$$= \frac{x}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi}{L} (mx + ny)} e^{-j q z}$$

Use (7) in (3) to find $E_{\text{array}}(x,y,z)$. 
The Green's function in (5) can be used to set up MOM solution to find actual current distribution, instead of assuming a form.

The solution begins as above, but \( J_{oo}(x,y) \) is represented in terms of basis functions with yet-to-be-determined amplitudes.

\[
J_{oo}(x,y) = \sum_{r=1}^{R} I_r B_r(x,y) \tag{8}
\]

where \( B_r(x,y) \) is an appropriate set of basis functions.

The resulting vector potential is

\[
A_{array}(x,y,z) = \frac{1}{4\pi} \sum_{r=1}^{R} I_r \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{i2\pi nm_d x} e^{-i2\pi nnd_y} \right\} \left( \sum_{unit cell} B_r(x',y') \frac{e^{i\frac{2\pi}{l}(x-x'-md_x)^2+(y-y'-nd_y)^2}+z^2}{\sqrt{(x-x'-md_x)^2+(y-y'-nd_y)^2+z^2}} dxdydz \right)
\]
Using (9) in (3) yields

\[ E_{\text{array}}(x, y, z) = \sum_{r=1}^{R} I_r \cdot E_r(x, y, z) \]  

(10)

where

\[ E_r(x, y, z) = -\overline{J}(x, y, z) + \]

\[ + \mu_\text{r} \times \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j\beta_m x} e^{-j\beta_n y} \]

\[ \sum_{\text{unit cell}} \bar{\mathbf{B}}_r(x', y') \frac{e^{-j\beta_m x'}}{\sqrt{\lambda}} dx'dy' \]

Suppose the antennas are strip dipoles with delta-gap voltage sources.
Evaluate (12) at \( z = 0 \) and require
\[
E_{\text{array}}(x, y, 0) = -E_{\text{inc}}(x, y, 0)
\]
\[
= \begin{cases} 
0 & \text{on strip} \\
-x \frac{V_0}{\delta} & \text{in gap}
\end{cases}
\]
where \( V_0 \) is the generator voltage.

As usual for MOM, (12) is enforced by testing with an appropriate set of functions. This generates a set of equations that can be solved for \( E I r^3 \). Once \( E I r^3 \) are determined, all antenna characteristics can be determined.

(Note: (12) is enforced only in the \((0,0)\) unit cell, and B.C. is then enforced in all unit cells by construction of the Green's function.)
Microstrip Dipole Array - Spectral Domain

\[ d_x = a \quad d_y = b \]

References:


a = b = 0.5 \lambda_0
\text{ \quad } d = 0.19 \lambda_0
\text{ \quad } \varepsilon_r = 2.55
\text{ \quad } l = 0.39 \lambda_0 \quad \text{\quad } w = 0.002 \lambda_0

\text{Pozar & Schaubert, IEEE T-AP, 1984}
Waveguide Simulators

Consider a (multibeam) phased array with two simultaneous excitations to produce beams at $\pm \theta_0$.

\[ V_{1, n} = V_0 e^{-j \frac{\pi}{2} n \omega d} \]

\[ V_{2, n} = V_0 e^{+j \frac{\pi}{2} n \omega d} \]

where \( u_0 = \sin \theta_0 \), \( \phi_0 = 0 \).

The radiated field is

\[ E = E_0 e^{-j \frac{\pi}{2} (1 - u_0^2) \omega d} \left[ e^{-j \frac{\pi}{2} u_0 x} + e^{+j \frac{\pi}{2} u_0 x} \right] 
= 2 E_0 e^{-j \frac{\pi}{2} (1 - u_0^2) \omega d} \cos(\frac{\pi}{2} u_0 x) \]
\[ E = 0 \quad \text{for} \quad x = \left( \frac{2m-1}{a} \right) \frac{\pi}{2 \pi} u_0 \]

Therefore, we can insert p.e.c. planes and not affect the array.

If the scan angle \( u_0 = \sin \theta_0 \) is adjusted so that center section and images are identical to original infinite array, then center section with p.e.c. walls behaves exactly like infinite array when it radiates two beams, \( \pm \theta_0 \).

Note: Requires element symmetry!!
Note: If $E_{tan} = 0$ at $y = \text{constant}$.

Now look at antenna voltages & currents.

$$V_n = V_{1, n} + V_{2, n} = V_0 \left[ e^{-j2u_0 n d} + e^{j2u_0 n d} \right]$$

$$= 2V_0 \cos(2u_0 n d) \quad (4)$$

Suppose we adjust $u_0$ so the null planes occur at $X = \pm d$

$$x = \pm d = \frac{2m + 1}{2} \pi \frac{1}{2u_0}$$

$$2u_0d = \pm \frac{\pi}{2}$$
Therefore,

\[ V_1 = V_{1,1} + V_{2,1} = 2V_0 \cos\left(\frac{\pi}{2}\right) = 0 \]

\[ V_0 = V_{1,0} + V_{2,0} = 2V_0 \cos(0) = 2V_0 \]

\[ V_4 = V_{1,4} + V_{2,4} = 2V_0 \cos\left(-\frac{\pi}{2}\right) = 0 \]

That is, the correct excitation to scan the beam to \( \Theta_0 \) such that \( E = 0 \) at \( x = \pm d \) requires \( V_{1,1} = -V_{2,1} \), etc. The elements at \( x = \pm d \) are effectively terminated in short circuit.
$W = 2d = \frac{\pi}{\theta_0} \Rightarrow \theta_0 = \sin^{-1} \frac{\pi}{\lambda_c W}$

That is,

$\sin \theta_0 = \frac{\lambda}{\lambda_c}$ (5)

where $\lambda_c = 2W =$ cut-off wavelength of $TE_{10}$ mode.

Note: The simulator condition (5) links frequency, scan angle and array element spacing. Changing frequency changes $\theta_0$. 
Look at input impedance of NCO element inside simulator.

\[ 2V_0 = V_{1,0} + V_{2,0} = I_{1,0} Z(\sin \theta_0) + I_{2,0} Z(\sin \theta_0) \]

- Total voltage of element
- Equal because array & elements are symmetric

\[ = 2I_0 \frac{Z(\sin \theta_0)}{Z(\sin \theta_0)} \]

Total current

\[ \frac{V}{I} = \frac{2V_0}{2I_0} = \frac{2I_0 Z(\sin \theta_0)}{2I_0} \]

\[ = Z(\sin \theta_0) \]

Same as infinite array radiating one beam at angle \( \theta_0 \)!!
Notes:

1. H-plane scan can be simulated by $\text{TE}_m$ modes in rectangular W/G.

2. $\text{TE}_m$ simulators apply only to linearly polarized elements.

3. Cannot simulate E-plane scan with p.e.c. simulator.

4. "Looking in" simulator. So far we have considered "looking out."

For lossless, reciprocal antennas, $|P_{in}| = |P_{out}|$. 

Antenna with load
Waveguide Simulator References


Fig. 6  Simulators for near-broadside (H-polarization).

Fig. 7  Simulators for far-from-broadside (H-polarization).
Fig. 8 Comparison of rectangular and triangular waveguides for E polarization (TM modes).

(a) SQUARE GRID.

(b) TRIANGULAR GRID.

Fig. 10 Useful one-port simulators (H-polarization).

Wheeler, Phased Array Antennas, 1972
Fig. 11. Measured and calculated reflection coefficient magnitude of a microstrip array in a waveguide simulator. Two patches are fed.