Principles of special relativity

- **Introduction.**
  Introducing optical devices, we saw that the electron-photon perturbation Hamiltonian is given by (see Lecture Notes, page 113, Eq. (179); page 265, Eq. (667)):

$$H_{phot} = \frac{ie\hbar}{mc}\mathbf{A}\cdot\nabla \ . \tag{A1}$$

  This interaction term can be obtained by starting from the free-electron Hamiltonian $\mathbf{p}^2/(2m)$ and replacing the electron momentum $\mathbf{p}$ with $\mathbf{p} - e\mathbf{A}/c$. (The factor of c in the denominators appears when using Gaussian units, which are more convenient in this context and we shall use them here). The reason behind this substitution relies on some basic principles of special relativity. Let's see how.

- **Galilean invariance and Maxwell's equations.**
  A soon as Maxwell's equation were formulated, it became clear that there was a major difference with respect to Newton's law. Let's start with Newton's second law,

$$\mathbf{F} = m\mathbf{a} \qquad \text{or} \qquad \mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2} \ . \tag{A2}$$

  and consider this same equation as could be written by somebody (called an 'observer') which is moving with respect to us with uniform velocity $\mathbf{u}$. Let's assume that we and the other observer use a reference frame with parallel x, y, and z axes, that the other observer moves along the x axis, so that $\mathbf{u} = (u, 0, 0)$, and that the origin of the two reference frames coincide at a fixed instant in time which we take $t = 0$. Then, calling (x, y, z) our frame and (x', y', z') the other observer's frame, an object located at a point (x, y, z) in our frame will be located at a point (x', y', z') in the observer's frame, such that:

$$\begin{aligned} x' &= x - ut \\ y' &= y \\ z' &= z \end{aligned} \ . \tag{A3}$$

Therefore, expressing Newton's second law in the 'primed' frame,

$$\mathbf{F} \;=\; m\frac{d^2\mathbf{x}'}{dt^2} \;=\; m\frac{d^2(\mathbf{x}-\mathbf{u}t)}{dt^2} \;=\; m\frac{d^2\mathbf{x}}{dt^2} \;-\; \frac{d\mathbf{u}}{dt} \;=\; m\frac{d^2\mathbf{x}}{dt^2}\;. \tag{A4}$$

In other words, Newton's law retains the same algebraic form in all frames which are moving with uniform velocity with respect to our frame. This principle can be generalized by saying that the laws of mechanics are valid in all 'inertial frames'. An observer, by performing experiments, cannot tell whether he/she is moving with respect to other inertial frames. This is the principle of Galilean relativity, since it was proposed (noted, discovered, invented?... the choice of a term is a matter of deep philosophical discussions) by Galileo.
Consider now Maxwell's equations. For simplicity, let's just consider a wave equation:

$$\left(\nabla^2 \;-\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\;\psi =\; 0\;. \tag{A5}$$

Let's apply the transformation Eq. (A3). Since in the 'unprimed' frame $\psi(x,y,z,t) = \psi(x'+ut,y,z,t)$, so that $\partial\psi/\partial x' = \partial\psi/\partial x + (1/u)\partial\psi/\partial t$, we find:

$$\left(\nabla^2 \;-\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \;-\; \frac{2u}{c^2}\frac{\partial^2}{\partial x\partial t} \;-\; \frac{u^2}{c^2}\frac{\partial^2}{\partial x^2}\right)\;\psi =\; 0\;. \tag{A6}$$

What a mess! The form of the equation has been completely altered by the transformation! In hindsight, we already knew it had to be so: After all, magnetic fields caused by moving charges must disappear when we use a frame in which the charges are at rest. Therefore, the $\mathbf{E}$ and $\mathbf{B}$ fields do not transform correctly under the Galilean transformation Eq. (A1). Moreover, the Lorentz force depends explicitly on the velocity of the particle, so that the form of the equation will differ in a different inertial frame. Historically, this is also related to the difficulty of understanding electromagnetic waves: Sound waves are oscillations of the medium in which they propagate. But electromagnetic waves are oscillations of what?

In order to fix the situation we have three alternatives we can choose from:

1. Maxwell's equations are wrong. The correct equations, yet to be discovered, are invariant under Galilean transformations.
2. Galilean invariance is valid for mechanics, not for electromagnetism. This is the historical solution before Einstein: The 'ether' determines the existence of the 'absolute frame' in which the ether is at rest and Mawxwell's equations hold.
3. Galilean invariance is wrong. There is a more general invariance – yet to be dsicovered – which preserves the form of Maxwell's equation. Classical mechanics is incorrect and must be reformulated so that it is invariant under this new transformation.

Having to chose between thrashing Maxwell (option 1) or Newton (option 3), physicists chose the easier option 2. Einstein, instead, decided to follow the third option, guided by two postulates:

1. **Postulate of relativity:** All physical laws must 'look the same' in all frames moving with uniform velocity with repect to each other.
2. **Postulate of the constancy of the speed of light:** The speed of light is the same (numerically the same!) independent of the velocity of the observer or of its source. This stems logically from the Michelson-Morley experiment of 1887, but the result could have been explained 'saving' ether and using the Lorentz-FitzGerald contractions.

Armed with these postulates, Einstein set to build a new set of transformations between inertial frames. Maxwell's equation are now invariant under this new set of transformations, but Newtonian mechanics has to be modified: If two frames move at a relative speed much smaller than the speed of light, the 'new' transformations approach the usual Galilean transfomation and Newton is approximately correct. But for relative velocities approaching the speed of light, the laws of mechanics deviate enormously from Newton's laws.

It is impossible to pay justice to special relativity in such a short time. We shall only discuss those few concepts which are required to answer our original question of why we perform the substitution $\mathbf{p} \rightarrow \mathbf{p} - (e/c)\mathbf{A}$.

- **Lorentz transformations.**

Consider the same two frames ('primed' K' and 'unprimed' K frames) considered above. Assume now that at $t = 0$ (the time at which the origins of the two frames coincided... at least when looking at our clock...!) a ray of light was emitted from the origin. Since light travels at the same speed in both frames, we must have for the wavefronts of the emitted light:

$$ct^2 - x^2 - y^2 - z^2 = 0 \quad \text{and} \quad ct'^2 - x'^2 - y'^2 - z'^2 = 0 \,, \tag{A7}$$

so that, assuming that space-time is isotropic and homogeneous,

$$ct^2 - x^2 - y^2 - z^2 = \lambda(u)[ct'^2 - x'^2 - y'^2 - z'^2] \,. \tag{A8}$$

The function $\lambda(u)$ is a possible velocity-dependent change of scale between the two frames. However, since going from K to K' must be involve the same transformation as going from K' to K with a sign-flip for u, we must have $\lambda = 1$. Using the notation $x_0 = ct$, $x_1 = x$, $x_2 = y$, and $x_3 = z$, Eq. (A8) is satisfied if:

$$\begin{array}{lcl} x_0' & = & \gamma(x_0 - \beta x_1) \\ x_1' & = & \gamma(x_1 - \beta x_0) \\ x_2' & = & x_2 \\ x_3' & = & x_3 \end{array} \,, \tag{A9}$$

where

$$\beta = \frac{u}{c} \quad \text{and} \quad \gamma = (1 - \beta^2)^{-1/2} \,. \tag{A10}$$

The transformations Eq. (A9) are called 'Lorentz transformations'. Obviously the inverse transformations read:

$$\begin{array}{lcl} x_0 & = & \gamma(x_0' + \beta x_1') \\ x_1 & = & \gamma(x_1' + \beta x_0') \\ x_2 & = & x_2' \\ x_3 & = & x_3' \end{array} \,, \tag{A11}$$

Note that, unlike the Galilean transformations, now time and space transform together: Simultaneous events in one frame will not be simultaneous in another frame. Indeed consider two events $(ct_1, \mathbf{x}_1)$ and $(ct_2, \mathbf{x}_2)$.

Thanks to Eq. (A8), their 'distance'

$$s_{12} = c^2(t_1 - t_2)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2 \tag{A12}$$

is an 'invariant' (that is, it's the same in all inertial frames). If $s_{12} > 0$ the separation between the events is said to be 'time-like': It is always possible to find a transformation (actually with $\beta = |\mathbf{x}_1 - \mathbf{x}_2|/(c|t_1 - t_2|)$) such that in the transformed frame the two events are at the same spatial location, separated only by time. If $s_{12} < 0$ the separation between the events is said to be 'space-like': It is always possible to find a Lorentz transformation such that in the transformed frame the two events are simultaneous, separated only spatially. If, finally, $s_{12} = 0$, the separation is said to be 'light-like': One event lies on the 'light-cone' of the other.

- **Proper time and time dilatation.**
Since 'time' has become an observer-dependent quantity, when dealing with moving particles it is convenient to consider the time in the frame in which the particle is at rest. If $\mathbf{v}(t)$ is the velocity of the moving particle in 'our' frame, let's consider the invariant:

$$ds^2 = dx_\mu dx^\mu = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt^2 - v^2dt^2 = c^2(1 - v^2/c^2)dt^2 = c^2dt^2/\gamma^2 . \tag{A13}$$

Since this quantity is invariant (that is, it is numerically the same in all inertial frames), in the frame in which the particle is at rest it will be $ds^2 = c^2d\tau^2$, where τ is the time in that frame. This time is called 'proper time'. If the particle travels over a time interval $\tau_2 - \tau_1$ in its proper time, as seen by us the time interval will stretch to the interval $t_2 - t_1$ obtained by integrating Eq. (A13) along the particle trajectory:

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} d\tau \ \gamma(\tau) = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1 - v(\tau)^2/c^2}} . \tag{A14}$$

Since $\gamma > 1$, the time interval we observe, $t_2 - t_1$, is longer than the proper time interval $\tau_2 - \tau_1$. This has been experimentally verified: The μ-mesons (actually, leptons) produced by cosmic-ray hits in the upper atmosphere, often reach the ground. Since the lifetime of the μ-meson is about 2.2 μs, even at the speed of light the particle could not travel more than about 660 m before decaying. Yet, they can easily be detected after having traveled distances more than two orders of magnitude longer (the thickness of the atmosphere, of the order of 10^5 m). This is because their lifetime, as observed by us, is stretched enormously, as these particles travel at speeds approaching the speed of light.

- **Lorentz contraction**.
  Consider a rod of length L' at rest in the K' frame. Let the ends of the rod be at $x' = x'_1$ and $x' = x'_2$, so that $L' = x'_2 - x'_1$. What is the length of the rod in our K frame? By the Lorentz transformations, Eq. (A9) (which we must use since when we measure the length of the rod we measure its ends at the same time in our frame), we have:

$$L = x_2 - x_1 = \frac{1}{\gamma}(x'_2 - x'_1) = \frac{L'}{\gamma} < L' \, . \tag{A15}$$

  The rod in our frame appears shorter than in its rest frame. This is the Lorentz-FitzGerald contraction which was postulated (without proof or arguments behind) in order to explain the Michelson-Morley experiment.

- **Addition of velocities.**
  Let's consider the Lorentz transformation

$$\begin{aligned} dx_0 &= \gamma(dx'_0 + \beta dx'_1) \\ dx_1 &= \gamma(dx'_1 + \beta dx'_0) \\ dx_2 &= dx'_2 \\ dx_3 &= dx'_3 \end{aligned} \, . \tag{A16}$$

  In our K frame, the velocity $v = c dx_1/dx_0$ of a particle moving with velocity $v' = c dx'_1/dx'_0$ in the K' frame will be:

$$v = c\,\frac{dx_1}{dx_0} = c\,\frac{dx'_1 + \beta dx'_0}{dx'_0 + \beta dx'_1} = c\,\frac{dx'_0(dx'_1/dx'_0 + \beta)}{dx'_0(1 + \beta dx'_1/dx'_0)} = \frac{v' + u}{1 + v'u/c^2} \, . \tag{A17}$$

  Note how the particle velocity v' and the frame velocity u add, as seen by us: In the limit of small velocities, $v \approx v' + u$, as usual. But as either v' or u approach c, the denominator grows and the sum-velocity v as seen

by us cannot exceed c. This is consistent with the second postulate.
Note that the 4-vector

$$U^\mu = (\gamma c, \gamma \mathbf{u}) \tag{A18}$$

transforms like the coordinate vector x^μ.

- **4-momentum.**
In classical mechanics the momentum and energy of a particle are:

$$\begin{aligned} E &= E(0) + \tfrac{1}{2}mu^2 \\ \mathbf{p} &= m\mathbf{u} \end{aligned} \quad . \tag{A19}$$

The term $E(0)$ is a constant which refers to the rest-energy of the particle. It is usually ignored in non-relativistic discussions.
In order to generalize these concepts, we can generally start with:

$$\begin{aligned} E &= \mathcal{E}(u) \\ \mathbf{p} &= \mathcal{M}(u)\mathbf{u} \end{aligned} \quad , \tag{A20}$$

with the constrains (dictacted by the fact that we want to recover Eq. (A19) in the limit $u \to 0$):

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial u^2}(0) &= \frac{m}{2} \\ \mathcal{M}(0) &= m \end{aligned} \quad . \tag{A21}$$

The general expressions for the functions $\mathcal{E}$ and $\mathcal{M}$ can be obtained by analyzing the elastic collision of two identical particles and require momentum and energy conservation in two inertial frames K and K'. We'll skip the derivation (see Jackson, *Classical Electrodynamics*, Sec. 11.5) and simply state the result: The general form for these two functions conistent with the two postulates, and energy and momentum conservation is:

$$\begin{aligned} \mathcal{E}(u) &= \gamma mc^2 \\ \mathcal{M}(u) &= \gamma m \end{aligned} \quad . \tag{A22}$$

Note that in the limit $u \to 0$, $E = \gamma mc^2 \;\to mc^2 + mu^2/2$, which is the classical result with a rest energy mc^2.

From this we can define an energy-momentum 4-vector, p^μ:

$$p^\mu \;=\; (\gamma mc, \gamma m\mathbf{u}) \;=\; (E/c, \gamma m\mathbf{u}) \;=\; mU^\mu \;, \tag{A23}$$

having used Eq. (A18) in the last step.

The invariant length of the energy-momentum 4-vector is

$$p^\mu p_\mu \;=\; p_0^2 \;-\; |\mathbf{p}|^2 \;=\; E^2/c^2 - \gamma^2 m^2 u^2 \;=\; \gamma^2 (mc^2 - mu^2) \;=\; mc^2 \;. \tag{A24}$$

Finally, from this expression we can write the energy E of a particle as:

$$E \;=\; \sqrt{c^2 p^2 + m^2 c^4} \;. \tag{A25}$$

- **Lorentz force.**

  Our goal now is to re-express the Lorentz force (recall that we are using Gassian units here):

$$\frac{d\mathbf{p}}{dt} \;=\; e(\mathbf{E} \;+\; \frac{\mathbf{u}}{c} \times \mathbf{B}) \tag{A26}$$

  in a manifetsly covariant form and find a Hamiltonian from which Eq. (A26) may be derived.

  First, let's re-write Eq. (A26) in terms of the proper time τ and add to it the equation expressing the power-balance $dE/d\tau = (e/c)\mathbf{u}\cdot\mathbf{E}$, so that we can employ the 4-vector p^μ:

$$\begin{aligned} \frac{d\mathbf{p}}{d\tau} \;&=\; \gamma\frac{d\mathbf{p}}{dt} \;=\; e\gamma\mathbf{E} \;+\; e\gamma\frac{\mathbf{u}}{c} \times \mathbf{B} \;=\; \frac{e}{c}(U^0 \;+\; \mathbf{U} \times \mathbf{B}) \\ \frac{dp^0}{d\tau} \;&=\; \gamma\frac{dp^0}{dt} \;=\; \gamma\frac{e}{c}\mathbf{u}\cdot\mathbf{E} \;=\; \frac{e}{c}\mathbf{U}\cdot\mathbf{E} \end{aligned} \;. \tag{A27}$$

Defining the electromagnetic field tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} , \tag{A28}$$

we can write Eq. (A27) in the manifestly covariant form:

$$\frac{dp^{\mu}}{d\tau} = \frac{e}{c} F^{\mu\nu} U_{\nu} \tag{A29}$$

We must now find a Hamiltonian function whose dynamic equations (the Hamilton equations of motion) yield the Lorentz-force equation. To do this correctly it would be necessary to develop a bit of Lagrangian theory. So, here we follow a 'pragmatic approach': Let's define the Hamiltonian:

$$H = \sqrt{c^2[c\mathbf{p} - (e/c)\mathbf{A}]^2 + m^2c^4} + e\Phi , \tag{A30}$$

where Φ is the scalar potential. Considering that the particle velocity in terms of $\mathbf{p}$ and $\mathbf{A}$ is (as it follows from Lagrangian theory):

$$\mathbf{u} = \frac{c\mathbf{p} - e\mathbf{A}}{\sqrt{(\mathbf{p} - (\mathbf{e}/\mathbf{cA})^2 + m^2c^4}} , \tag{A31}$$

with some algebra one can verify that the Hamilton equations of motion ($i = 1, 2, 3$):

$$\frac{\partial H}{\partial x_i} = \frac{dp^i}{d\tau} \tag{A32}$$

is equivalent to the Lorentz-force equation (the first of the two Eqns. (A27)). Comparing this expression with Eq. (A25) we see that the interaction between a charged particle and the electromagnetic field has been accounted for by replacing the particle momentum $\mathbf{p}$ with $\mathbf{p} - (e/c)\mathbf{A}$, which is what we wanted to show.