Problem #1:	_______
2:	_______
3:	_______
4:	_______
5:	_______
6:	_______
Total:	_______

ECE 604 LINEAR SYSTEMS

Fall 2011

Final Exam

Issued: December 8, 2011 Due: December 13, 2011

Consistent with the ECE Honor Code, you are asked to read the following voluntary statement carefully and sign it before beginning your work in each exam booklet:

I have not given or received unauthorized aid on this exam.

______________________________ Signature ____________

Instructions

The exam is a take-home exam. You should not discuss these problems with others. Do not use Matlab (or any equivalent program such as Mathematica) to generate your answers (you can use any computer program to check your answers).

There are six questions with the following values:

Problem #1	15	points
Problem #2	15	points
Problem #3	15	points
Problem #4	15	points
Problem #5	15	points
Problem #6	25	points
Total	100	points

Problem 1: (15 points) An Earth satellite of unit mass can be modeled as a point mass moving in a plane while attracted to the origin of the plane by an inverse square law force. It is convenient to choose polar coordinates, with $r(t)$ the radius from the origin to the mass, and $\theta(t)$ the angle from an appropriate axis. Assuming the satellite can apply forces $u_1(t)$ in the radial direction and $u_2(t)$ in the tangential direction (see Figure 1), the equations of motion have the form:

$$
\begin{aligned}
\ddot{r}(t) &= r(t)\dot{\theta}^2(t) - \frac{\beta}{r^2(t)} + u_1(t) \\
\ddot{\theta}(t) &= -\frac{2\dot{r}(t)\dot{\theta}(t)}{r(t)} + \frac{u_2(t)}{r(t)}
\end{aligned}
\tag{1}
$$

where β is a constant.

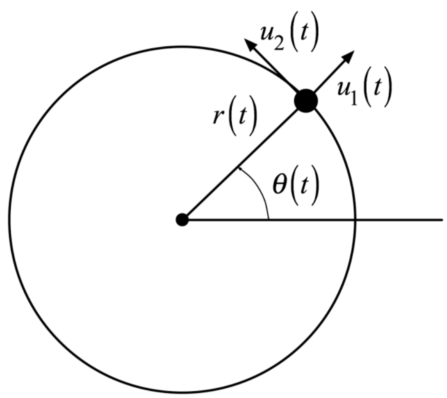

Figure 1

a) When the thrust forces are zero, the solutions to these equations can be ellipses, parabolas or hyperbolas. The simplest solution is a circle with both $r(t) = r_0$ and $\dot{\theta}(t) = \omega_0$ constant. Given ω_0, find the nominal radius r_0.
b) Using the state vector $\mathbf{x}^T(t) = \begin{bmatrix} r(t) & \dot{r}(t) & \theta(t) & \dot{\theta}(t) \end{bmatrix}$, the thrust forces as the inputs, and the radius and angle as the outputs, write the equations of motion as a (nonlinear) state variable model.
c) Find the nominal state solution for the trajectory from a).
d) Linearize the state model around the nominal solution from c). Clearly define the states, inputs and outputs for the linear state model.

Problem 2: (15 points) Consider the linear system that is at rest for all negative times:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t) \qquad \mathbf{x}(0)=0\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)\end{aligned}$$

The input to this system only changes values at *sample* times $\{kT : k = 0,1,2,\ldots\}$ and is constant between these samples[1]:

$$\mathbf{u}(t)=\begin{cases}\tilde{\mathbf{u}}_k & kT \le t < (k+1)T,\ k=0,1,2,\ldots\\ 0 & \text{otherwise}\end{cases}$$

The measurement of the linear system is synchronously sampled and is the average of the continuous-time output:

$$\tilde{\mathbf{y}}_k=\begin{cases}\dfrac{1}{T}\displaystyle\int_{(k-1)T}^{kT}\mathbf{y}(\sigma)\,d\sigma & k=0,1,2,\ldots\\ 0 & \text{otherwise}\end{cases}$$

A linear, time-invariant, discrete-time state variable model has the form

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1} &= \mathbf{F}\tilde{\mathbf{x}}_k+\mathbf{G}\tilde{\mathbf{u}}_k \qquad \tilde{\mathbf{x}}_0 \text{ given}\\ \tilde{\mathbf{y}}_k &= \mathbf{H}\tilde{\mathbf{x}}_k+\mathbf{J}\tilde{\mathbf{u}}_k\end{aligned}$$

Find **F**, **G**, **H,** and J in terms of the continuous-time linear system matrices **A**, **B**, **C**, and **D**.

[1] That is, $\mathbf{u}(t)$ could be the output of an ideal ZOH.

Problem 3: (15 points) Consider the two linear systems

$$\dot{\mathbf{x}}_1(t) = \begin{bmatrix} 0 & 1 \\ 2-t^2 & 2t \end{bmatrix} \mathbf{x}_1(t)$$

$$\dot{\mathbf{x}}_2(t) = \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix} \mathbf{x}_2(t)$$

Both systems are state variable representations of the differential equation

$$\ddot{y}(t) - 2t\dot{y}(t) - \left(2-t^2\right)y(t) = 0$$

a) Show these two state equations are equivalent by finding a state transformation

$$\mathbf{x}_2(t) = \mathbf{P}(t)\mathbf{x}_1(t)$$

Note that you must show $\mathbf{P}(t)$ this is a state transformation.

Hint: Define the components of the two state vectors as:

$$\mathbf{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} \qquad \mathbf{x}_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}$$

Take the derivative of the first equation for $\mathbf{x}_1(t)$:

$$\begin{aligned} \ddot{x}_{11}(t) = \dot{x}_{12}(t) &= \left(2-t^2\right)x_{11}(t) + 2tx_{12}(t) \\ &= \left(2-t^2\right)x_{11}(t) + 2t\dot{x}_{11}(t) \end{aligned}$$

Thus, $x_{11}(t)$ satisfies the same 2nd order differential equation as $y(t)$. Hence:

$$\mathbf{x}_1(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

Now differentiate the first equation for $\mathbf{x}_2(t)$:

$$\begin{aligned} \ddot{x}_{21}(t) = x_{21}(t) + t\dot{x}_{21} + \dot{x}_{22}(t) &= x_{21}(t) + t\dot{x}_{21} + \left(x_{21}(t) + tx_{22}(t)\right) \\ &= 2x_{21}(t) + t\dot{x}_{21}(t) + t\left(\dot{x}_{21}(t) - tx_{11}(t)\right) \\ &= \left(2-t^2\right)x_{21}(t) + 2t\dot{x}_{21}(t) \end{aligned}$$

Thus, $x_{21}(t)$ satisfies the same 2nd order differential equation as $y(t)$. Also, from the first equation for $\mathbf{x}_2(t)$:

$$x_{22}(t) = \dot{x}_{11}(t) - tx_{11}(t) = \dot{y}(t) - ty(t)$$

Hence:

$$\mathbf{x}_2(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) - ty(t) \end{bmatrix}$$

b) Find the state transition matrix for the system with state vector $\mathbf{x}_2(t)$.

Hint: The system matrix for this equation can be written as:

$$\begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix} = t\mathbf{I} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \mathbf{M} + \mathbf{N}$$

The matrices **M** and **N** commute.

c) Find the state transition matrix for the system with state vector $\mathbf{x}_1(t)$.

Problem 4: (15 points) Consider the transfer function

$$G(s) = \begin{bmatrix} \dfrac{s+1}{s^2+2s+1} & \dfrac{s}{s^2+1} \\ \dfrac{1}{s+2} & \dfrac{2}{s^2+3s+2} \end{bmatrix}$$

You may use computer programs for this problem.

a) Find the first eight Markov parameters for $G(s)$ (i.e., H_0 through H_7).

b) What is the order of the minimal realization for $G(s)$?

c) Find a minimal realization for $G(s)$ using its Markov parameters.

Problem 5: (15 points) Consider the state variable model of a dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{u}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

where $(\mathbf{A}, \mathbf{C})$ is observable. The system is to be stabilized using constant gain output feedback:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{y}(t)$$

The constant gain $\mathbf{K}$ will be defined in terms of the matrix:

$$\mathbf{M}_L = \int_0^L e^{-\mathbf{A}^{\mathrm{T}}\sigma}\mathbf{C}^{\mathrm{T}}\mathbf{C}e^{-\mathbf{A}\sigma}\,d\sigma$$

where L>0.

a) Show that $\mathbf{M}_L$ is invertible.

b) Show that

$$\mathbf{A}^{\mathrm{T}}\mathbf{M}_L + \mathbf{M}_L\mathbf{A} = \mathbf{C}^{\mathrm{T}}\mathbf{C} - e^{-\mathbf{A}^{\mathrm{T}}L}\mathbf{C}^{\mathrm{T}}\mathbf{C}e^{-\mathbf{A}L}$$

Hint: Show that the left side is an integral of a perfect derivative.

c) Show that

$$\mathbf{u}(t) = -\mathbf{M}_L^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{y}(t)$$

results in an exponentially stable closed-loop system.

Hint: Show that $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{M}_L\mathbf{x}$ is a Lyapunov function for the closed-loop system. Apply the invariance principle to obtain exponential stability.

Problem 6: (25 points) Consider the system:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

a) Using state feedback, place the poles of the closed loop system at $\{-1,-1,-1\}$

b) Specify an observer for this system that has poles at $\{-2,-2,-2\}$.

c) What is the estimation error $\mathbf{e}$ (define the error in terms of the original system and the estimator, and specify the differential equation it satisfies)?

d) What is the compensator that results using the state feedback gain of part a and the observer of part b?

e) What are the error dynamics of the system? (The error state should include the original system states and the estimation error.)