Introduction

• Many problems can be formulated as maximizing or minimizing an objective, under limited resources and constraints

• If
  – Objective can be specified as linear function of certain variables
  – Constraints on resources can be specified as equalities or inequalities on those variables

• We have a linear programming problem
Example: Political Problem ;-)

- Assume you are a politician (quite a change to becoming an electrical engineer!!) trying to win an election
- You district consists of an urban, suburban, and rural area of 100,000, 200,000, and 50,000 registered voters respectively
- You would like to win the majority in all areas
- You are aware that advertising on certain issues might help you win the election

The issues of concern are:
- Building more roads
- Gun control
- Farm subsidies
- Gasoline tax

You can estimate how many votes you will win or lose from each population segment by spending $1,000 on advertising on each issue.

<table>
<thead>
<tr>
<th>Policy</th>
<th>Urban</th>
<th>Suburban</th>
<th>Rural</th>
</tr>
</thead>
<tbody>
<tr>
<td>Build roads</td>
<td>-2</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Gun control</td>
<td>8</td>
<td>2</td>
<td>-5</td>
</tr>
<tr>
<td>Farm subsidies</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Gasoline tax</td>
<td>10</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>
Example: Political Problem ;-) 

• Your task is to figure out the minimum amount of money to be spend in order to win 50,000 urban, 100,000 suburban, and 25,000 rural votes

• This can be approached by trial and error
  – Strategy may not be least expensive one:
    – E.g.: $20,000 on building roads, $0 on gun control, $4,000 on farm subsidies, and $9,000 on gasoline tax
    – You would win:
      • 20(-2) + 0(8) + 4(0) + 9(10) = 50k urban votes
      • 20(5) + 0(2) + 4(0) + 9(0) = 100k suburban votes
      • 20(3) + 0(-5) + 4(10) + 9(-2) = 82k rural votes

Example: Political Problem ;-) 

• This way you would win the exact number of votes in urban and suburban areas and more than enough (and even more than actual voters) in the rural area

• This would come at a cost of 20 + 0 + 4 + 9 = $33k for advertising

• Best strategy possible?
  – Additional trial and error might help
  – What about a systematic method?
• Let’s introduce 4 variables:
  – $X_1$ is the number of thousands of dollars spent on ads on building roads,
  – $X_2$ is the number of thousands of dollars spent on ads on gun control,
  – $X_3$ is the number of thousands of dollars spent on ads on farm subsidies,
  – $X_4$ is the number of thousands of dollars spent on ads on gasoline tax.
• Requirement to win at least 50,000 urban votes:
  – $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
• Requirements for suburban and rural votes:
  – $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$
  – $3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

Example: Political Problem ;-)
• In addition minimize $x_1 + x_2 + x_3 + x_4$ with $x_1 \geq 0,x_2 \geq 0,x_3 \geq 0,x_4 \geq 0$
• Now we can formulate linear programming problem as:
  • Minimize $x_1 + x_2 + x_3 + x_4$
  • Subject to
    
    \[
    \begin{align*}
    -2x_1 + 8x_2 + 0x_3 + 10x_4 & \geq 50 \\
    5x_1 + 2x_2 + 0x_3 + 0x_4 & \geq 100 \\
    3x_1 - 5x_2 + 10x_3 - 2x_4 & \geq 25 \\
    x_1, x_2, x_3, x_4 & \geq 0
    \end{align*}
    \]
  – Solution of this linear program will yield an optimal strategy!
General Linear Programs

- Optimize linear function subject to a set of linear inequalities
- Given a set or real numbers $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function $f$ on those variables is defined by:
  - $f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{j=1}^{n} a_n x_n$
  - If $b$ is a real number and $f$ is a linear function, then the equation $f(x_1, x_2, \ldots, x_n) = b$ is a linear equality and the inequalities $f(x_1, x_2, \ldots, x_n) \geq b$ and $f(x_1, x_2, \ldots, x_n) \leq b$ are linear inequalities

Linear Programming Overview

- Two forms, slack and standard
- Standard:
  - Maximization of linear function subject to linear inequalities
- Slack:
  - Maximization of linear function subject to linear equalities
- For now, maximizing of a linear function with $n$ variables subject to a set of $m$ linear inequalities
Let's consider a linear program with two variables:

- **Minimize** $x_1 + x_2$
- **Subject to**
  
  
  
  \[
  \begin{align*}
  4x_1 - x_2 & \leq 8 \\
  2x_1 + x_2 & \leq 10 \\
  5x_1 - 2x_2 & \geq -2 \\
  x_1, x_2 & \geq 0
  \end{align*}
  \]

  - All settings of variables $x_1$ and $x_2$ that satisfy all constraints are called **feasible solution** to the linear programming program.
  - Function we wish to optimize is called **objective function**.

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**Linear Programming Overview**

- Area surrounded by all constraints defines set of feasible solutions:
  - **Feasible region**
  - Particular value of objective function at particular point is called **objective value**
  - Feasible region contains an infinite number of points
  - Find efficient way to find point that achieves maximum objective value without explicitly evaluating objective function at every point
Visually, we can determine that $x_1 = 2$ and $x_2 = 6$ is the optimal solution.

Same intuition holds for cases with more than two variables.

For three variables, each constraint is defined by half-space and intersection of half-spaces defines feasible region.

Set for which objective function obtains a specific value is now a plane.

Because feasible region is convex, optimal objective value must contain a vertex of feasible region.

Similarly, in the case of $n$ variables, each constraint defines half-space in $n$-dimensional space.

Feasible region formed by intersection of these half-spaces is called **simplex**.

Objective function is now a hyperplane.

Because of convexity, solution will still occur at vertex of simplex.
Simplex Algorithm

- **Simplex algorithm** takes as input a linear program and returns an optimal solution.
- Starts at some vertex of the simplex and performs sequence of iterations.
- In each iteration, it moves on edge of simplex from current vertex to a neighboring one with no objective value smaller than the former one.
- Algorithm terminates when it reaches local maximum.
- Since feasible region is convex and objective function is linear => local optimum is actually a global optimum.

Applications of Linear Programming

- Has a large number of applications.
- One very popular area is Operations Research.
- Airline scheduling flight crews
  - Many FAA constraints: # of consecutive hours crew can work, work only on particular model of aircraft during one month.
  - Airline wants to schedule crews on all flights using as few crew members as possible.
- Oil company deciding where to drill for oil
  - Siting drill at particular location has associated cost based on geological surveys an expected payoff on number of barrels.
  - Limited budget for location new drills but wants to maximize of expected oil to find.
Algorithms for Linear Programming

• The simplex algorithm, when implemented carefully, often solves general linear programming problems quickly
• Unfortunately, with some inputs the simplex algorithm can require exponential time
• If we add the additional requirement that all variables take on integer values => integer linear program
• No know polynomial-time algorithm for this problem, since it is NP-hard.
• In contrast, general linear programming problem is solvable in polynomial time

Standard Form

• All constraints are inequalities (in slack form they are equalities)
• Given $n$ real numbers $c_1, c_2, ..., c_n$; $m$ real numbers $b_1, b_2, ..., b_m$; and $mn$ real numbers $a_{ij}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.
• We wish to find $n$ real numbers $x_1, x_2, ..., x_n$ that

  maximize $\sum_{j=1}^{n} c_j x_j$

  subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for $i = 1, 2, ..., m$

  $x_j \geq 0$ for $j = 1, 2, ..., n$. 
Standard Form

- Always possible to convert minimization or maximization of linear function into standard form

- Not in standard form because:
  1. Objective function might be minimization rather than maximization
  2. There may be variables without nonnegative constraints
  3. Equality constraints which have an equal sign rather than a less-than-or-equal-to sign
  4. Inequality constraints but rather than having a less-than-or-equal-to sign they have a greater-than-or-equal-to sign

Standard Form

- To convert minimization linear problem $L$ into an equivalent maximization linear problem $L'$
  - Simply negate the coefficients in the objective function

- Minimize $-2x_1 + 3x_2$
  Maximize $2x_1 - 3x_2$

Subject to

$x_1 + x_2 = 7$
$x_1 - 2x_2 \leq 4$
$x_1 \geq 0$
$x_1 + x_2 = 7$
$x_1 - 2x_2 \leq 4$
$x_1 \geq 0$
Standard Form

• Convert linear program in which variables do not have nonnegative constraints into one in which each variable has nonnegative constraint

• Suppose $x_j$ does not have nonnegative constraint
  – Replace each occurrence of $x_j$ by $x'_j - x''_j$
  – Add nonnegative constraints $x'_j \geq 0$ and $x''_j \geq 0$
  – If objective function has term $c_j x_j$ it is replaced by $c_j x'_j - c_j x''_j$, and
  – Term $a_{ij} x_j$ it is replaced by $a_{ij} x'_j - a_{ij} x''_j$

• Any feasible solution $\bar{x}$ to the new linear program corresponds to a feasible solution of original linear program with $\bar{x} = \bar{x}'_j - \bar{x}''_j$, with same objective value => two linear programs are equivalent

Standard Form

• Continuing example, ensure that each variable has a corresponding nonnegative constraint

• Variable $x_1$ has constraint but not $x_2$

• Replace $x_2$ by $x'_2$ and $x''_2$

• Modify linear program

• Maximize $2x_1 - 3x'_2 + 3x''_2$

  Subject to $x_1 + x'_2 - x''_2 = 7$
  $x_1 - 2x'_2 + 2x''_2 \leq 4$
  $x_1, x'_2, x''_2 \geq 0$
Next, convert equality constraints into inequality constraints

Suppose equality constraint \( f(x_1, x_2, \ldots, x_n) = b \)
- Since \( x = y \) if and only if \( x \leq y \) and \( x \geq y \)
- Replace constraint by pair of inequality constraints \( f(x_1, x_2, \ldots, x_n) \leq b \) and \( f(x_1, x_2, \ldots, x_n) \geq b \)
- Finally convert greater-than-or-equal-to constrain to less-than-or-equal-to by multiplying constraints by \(-1\)

Any inequality of the form \( \sum_{j=1}^{n} a_{ij} x_j \geq b_i \) is equivalent to \( \sum_{j=1}^{n} -a_{ij} x_j \leq -b_i \)

By replacing each coefficient \( a_{ij} \) by \(-a_{ij}\) and each value \( b_{ij} \) by \(-b_{ij}\), equivalent less-than-or-equal-to constraints are obtained

Finishing the example:

Maximize \[ 2x_1 - 3x_2' + 3x_2'' \]
Subject to \[ x_1 + x_2' - x_2'' \leq 7 \]
\[ x_1 + x_2' - x_2'' \geq 7 \]
\[ x_1 - 2x_2' + 2x_2'' \leq 4 \]
\[ x_1, x_2', x_2'' \geq 0 \]
Standard Form

• Finally, negating second constraint and renaming $x'_2$ to $x_2$ and $x''_2$ to $x_2$, results in the standard form:

• Maximize $2x_1 - 3x_2 + 3x_3$

Subject to

\[
\begin{align*}
  & x_1 + x_2 - x_3 \leq 7 \\
  & -x_1 - x_2 + x_3 \leq -7 \\
  & x_1 - 2x_2 + 2x_3 \leq 4 \\
  & x_1, x_2, x_3 \geq 0
\end{align*}
\]

Slack Form

• To efficiently solve a linear program with simplex algorithm, prefer to extend it in form in which some of the constraints are equality constraints

• More precisely, convert it into a form in which nonnegative constraints are only inequality constraints, remaining constraints are equalities

• Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ be an inequality constraint

• Introduce variable $s$ and rewrite the inequality as two constraints

\[
\begin{align*}
  & s = b_i - \sum_{j=1}^{n} a_{ij}x_j \\
  & s \geq 0
\end{align*}
\]
Slack Form

- $s$ is called a **slack variable** because it measures slack (or difference) between the left-hand and right-hand side of $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$
- Because this inequality is only true if both $s = b_i - \sum_{j=1}^{n} a_{ij}x_j$ and $s \geq 0$ are true => conversion can be applied to each inequality constraint of a linear program
- This results in a linear program where only inequality constraints are nonnegative ones
- Use $x_{n+i}$ instead of $s$ to denote slack variable associated with the $i$th inequality $x_{n+i} = b_i \sum_{j=1}^{n} a_{ij}x_j$, and constraint $x_{n+i} \geq 0$

Slack Form Example

- For problem from slide 24, we introduce slack variables $x_4$, $x_5$, and $x_6$ obtaining:

  Maximize $2x_1 - 3x_2 + 3x_3$

  Subject to $x_4 = 7 - x_1 - x_2 + x_3$
  $x_3 = -7 + x_1 + x_2 - x_3$
  $x_6 = 4 - x_1 + 2x_2 - 2x_3$
  $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$
Slack Form Example

- Omit “maximize”, “subject to”, and nonnegative contraints for linear program that satisfies these conditions
- Use variable $z$ to denote value of objective function
- Slack form of linear program:
  - $z = 2x_1 - 3x_2 + 3x_3$
  - $x_4 = 7 - x_1 - x_2 + x_3$
  - $x_3 = -7 + x_1 + x_2 - x_3$
  - $x_6 = 4 - x_1 + 2x_2 - 2x_3$

Slack Form: Concise Notation

- Use $N$ to denote set of indices of nonbasic variables
- Use $B$ to denote set of indices of basic variables
- Also $|N| = n$, $|B| = m$, and $N \cup B = \{1, 2, \ldots, n + m\}$
- As in standard form we use $b_i$, $c_j$, and $a_{ij}$ to denote constant terms and coefficients
- Use $v$ to denote optional constant term in objective function
  - $z = v + \sum_{j \in N} c_j x_j$
  - $x_i = b_i - \sum_{j \in N} a_{ij} x_j$
Slack Form: Concise Notation

• With slack form

\[
\begin{align*}
    z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
    x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{2x_6}{3} \\
    x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
    x_4 &= 18 - \frac{3x_3}{2} - \frac{x_5}{2} \\
\end{align*}
\]

Slack Form: Concise Notation

• We have \( B = \{1,2,4\}, N = \{3,5,6\} \)

• \( A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix} \)

• \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix} \)

• \( c = (c_3 \ c_5 \ c_6)^T = (-1/6 - 1/6 - 2/3)^T \), and \( v = 28 \).

• Indices into \( A, b, \) and \( c \) are not necessarily sets of contiguous integers; depend on index sets \( B \) and \( N \).
Problems as Linear Programs

• Shortest path can be formulated as linear program

• Focus on formulation of single-pair shortest path problem:
  – Weighted, directed graph $G = (V, E)$ with weight function $w : E \rightarrow R$ mapping edges to real-valued weights
  – A source vertex $s$, and a destination vertex $t$

• Goal is to calculate value $d[t]$, which is a weighted shortest path from $s$ to $t$.

Problems as Linear Programs

• Formulating this as linear program, need to determine a set of variables and constraints that define when there is a shortest path from $s$ to $t$.

• That is exactly what Bellman-Ford algorithm does

• When B-F finishes terminates, it has composed for each vertex $v$, a value $d[v]$, s.t. for each edge $(u,v) \in E$ we have $d[v] \leq d[u] + w(u,v)$

• Source vertex initially receives a value $d[s] = 0$, which never changes
Problems as Linear Programs

• This results in the following linear program to compute the shortest path weight from s to t:
  • Maximize \( d(t) \)
  • Subject to \( d(v) \leq d(u) + w(u, v) \) for each edge \((u, v) \in E\)
    \[ d(s) = 0 \]
  • There are \(|V|\) variables \(d(v)\), one for each vertex \(v \in V\)
  • There are \(|E| + 1\) constraints, one for each edge plus the additional one that the source vertex is always 0.

Simplex Algorithm

• Extended example of simplex algorithm
  • Maximize \( 3x_1 + x_2 + 2x_3 \)
  Subject to \( x_1 + x_2 + 3x_3 \leq 30 \)
    \( 2x_1 + 2x_2 + 5x_3 \leq 24 \)
    \( 4x_1 + x_2 + 5x_3 \leq 24 \)
    \( x_1, x_2, x_3 \geq 0 \)
Simplex Algorithm

• Convert linear program into slack form to use simplex algorithm

• Slack is also useful algorithmic concept:
  – Recalling that each variable has nonnegativity constraint, equality constraint is tight for a particular setting of its nonbasic variables if they causes the constraint's basic variable to become 0.
  – Setting of nonbasic variables that would make basic variable become negative violates that constraint
  – Slack variable explicitly maintains how far each constraint is from being tight
  – Help determine how much we can increase values of nonbasic variables without violating any constraints.

Simplex Algorithm

• Associating slack variables $x_4$, $x_5$, and $x_6$ with the inequalities and putting linear program into slack form:
  
  \[
  \begin{align*}
  z &= 3x_1 + x_2 + 2x_3 \\
  x_4 &= 30 - x_1 - x_2 - 3x_3 \\
  x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 &= 36 - 4x_1 - x_2 - 2x_3
  \end{align*}
  \]

• System has 3 equations and 6 variables. Any setting of variables $x_1$, $x_2$, and $x_3$ determines values for $x_4$, $x_5$, and $x_6$.

• Therefore, an infinite number of solutions for this system of equations exists.
Simplex Algorithm

• Goal: in each iteration, reformulate linear program such that the basic solution has greater objective value.
• Select non-basic variable $x_e$ whose coefficient in objective function is positive
• Increase value of $x_e$ as much as possible without violating constraints
• Variable $x_e$ becomes basic and some other variable $x_1$ becomes nonbasic
• Values of other basic variables and objective function may also change

Simplex Algorithm

• Continue example: Let’s increase value of $x_1$, which will decrease values $x_4$, $x_5$, and $x_6$
• Due to nonnegative constraint, none of them can become negative
• $x_1$ larger than 30, 12, and 9 respectively will break these constraints
• The last constraint is the tightest constraint, which limits how much $x_1$ can be increased
• Therefore switch roles of $x_1$ and $x_6$
Simplex Algorithm

- Solve last constraint from slide 38 for $x_1$ and obtain

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

- To rewrite other equations with $x_6$ on right-hand side, we substitute for $x_1$ using the equation above

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$= 30 - \left(9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}\right) - x_2 - 3x_3$$

$$= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$

Simplex Algorithm

- Replacing $x_1$ in the remaining constraints and the objective function results in:

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{x_6}{4}$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$

$$x_5 = 6 - \frac{3x_2}{4} - 4x_3 + \frac{x_6}{2}$$
• This operation is called a **pivot**.
  – As demonstrated above, a pivot chooses a nonbasic variable \( x_e \) called the **entering variable**, and a basic variable \( x_l \), called the **leaving variable**, and exchange their roles.

• The linear programs described on slides 42 and 38 are equivalent
  – Basic solution \((0, 0, 0, 30, 24, 36)\) satisfies new equations and has objective value \(27 + (1/4)\cdot0 + (1/2)\cdot0 – (3/4)\cdot36 = 0\)
  – Basic solution for new equations sets nonbasic values to 0 and is \((9, 0, 0, 21, 6, 0)\) with objective value \(z=27\).
  – This also satisfies the equations on slide 38.

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• Continue example: Wish to find new variable whose value we would like to increase
  
• Do not want to increase \( x_6 \), since increase in its value, value of objective function decreases

• Can attempt to increase either \( x_2 \) or \( x_3 \). We choose \( x_3 \)

• How far can we increase \( x_3 \) without violating constraints?

• First constraint limits it to 18, second to 42/5, and third to 3/2
Simplex Algorithm

• Again, third constraint is tightest one
• Rewrite third constraint such that $x_3$ is on left-hand side and $x_5$ on right-hand side

\[
\begin{align*}
    z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
    x_1 &= \frac{4}{3} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
    x_3 &= \frac{2}{8} - \frac{3x_2}{4} + \frac{x_5}{8} + \frac{x_6}{8} \\
    x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{align*}
\]

Simplex Algorithm

• This system has basic solution $(33/4, 9, 3/2, 69/4, 0, 0)$ with objective value $111/4$
• Now only way to increase objective value is to increase $x_2$
• Three constraints give upper bounds of 132, 4, and $\infty$
  – Upper bound of last constraint is $\infty$ because as $x_2$ increases so does $x_3$
  – Thus this constraint places no restriction on how much $x_2$ can increase
• $x_2 = 4$ makes third constraint tightest one
Simplex Algorithm

• Solve second constraint for $x_2$ and substitute in other equations

\[
\begin{align*}
  z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
  x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
  x_3 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
  x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{align*}
\]

Simplex Algorithm

• Now all coefficients in objective function are negative
• This only happens when basic solution is an optimal solution
• Thus the solution (8, 4, 0, 18, 0, 0) with objective value 28 is optimal
• Returning to original linear program on slide 38, only variables are $x_1, x_2, x_3$ and solution is $x_1 = 8, x_2 = 4, x_3 = 3$, with objective value 28
Next Steps

• Next lecture and on Thursday
• Project 2 due on Thursday