ECE 671 – Lecture 17

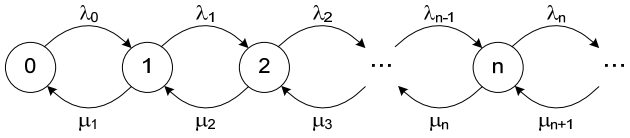
Queuing Theory
Steady-State Analysis

Queuing theory basics

- Express process as Markov chain
  - Discrete time
  - Continuous time
- Steady state probability
  - Probability distribution of states in the limit
- Next: use Markov chain to model queuing in network

Birth-Death Processes

- Solving general Markov chain can be difficult
- Simpler, constrained version: birth-death process
  - Transitions are only allowed between neighboring states
  - Transition rates: birth rate λ_k and death rate μ_k
- Birth-death process:



  - Matrix form:

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Steady State of Birth-Death Processes

- Steady state equations:
  - $0 = -\pi_0\lambda_0+\pi_1\mu_1$
  - $0 = -\pi_k(\lambda_k+\mu_k)+\pi_{k-1}\lambda_{k-1}+\pi_{k+1}\mu_{k+1}$
- Solving for π:
  - $\pi_1=\lambda_0/\mu_1\ \pi_0$
  - $\pi_2=\lambda_0\lambda_1/(\mu_1\mu_2)\ \pi_0$
  - In general: $\pi_k = \pi_0 \cdot \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}$, $k \geq 1$
- What about π_0?
  - Sum of probabilities must be 1

$$\pi_0 = \frac{1}{1+\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\frac{\lambda_i}{\mu_{i+1}}} = \frac{1}{\sum_{k=0}^{\infty}\prod_{i=0}^{k-1}\frac{\lambda_i}{\mu_{i+1}}}$$

  - Convergence criterion: $\exists k_0, \forall k>k_0: \lambda_k/\mu_k<1$

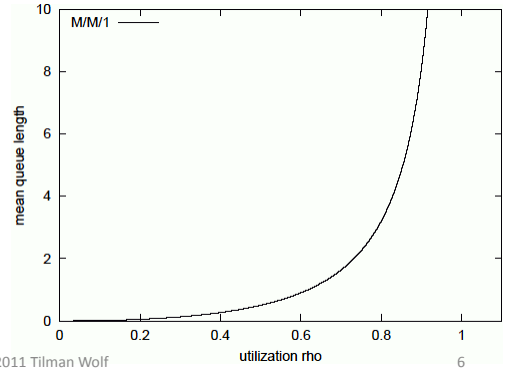
$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Birth-Death Process Example

- Simplest example
  - All birth rates are the same (=λ)
  - All death rates are the same (=μ)
- Solve π_0: $$\pi_0 = \frac{1}{1+\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\frac{\lambda}{\mu}} = \frac{1}{1+\sum_{k=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^k} = \frac{1}{1+\frac{\lambda/\mu}{1-\lambda/\mu}} = 1-\frac{\lambda}{\mu}$$
- Then π_k: $$\pi_k = \pi_0 \cdot \left(\frac{\lambda}{\mu}\right)^k = \left(1-\frac{\lambda}{\mu}\right)\cdot\left(\frac{\lambda}{\mu}\right)^k$$
- Represent utilization ρ=λ/μ
  - $\pi_k=(1-\rho)\rho^k$
- Geometric distribution (with parameter p=(1-ρ))

Birth-Death Process Example

- Mean number of customers in system:
  - $$\overline{N} = \sum_{k=1}^{\infty} k\cdot\pi_k = \sum_{k=1}^{\infty} k\cdot(1-\rho)\rho^k = \frac{\rho}{1-\rho}$$
- With Little's law:
  - T=N/λ=1/μ/(1-ρ)
  - Q=ρ^2/(1-ρ)
- So, finally:
  - With increasing load, queue length and waiting time increase

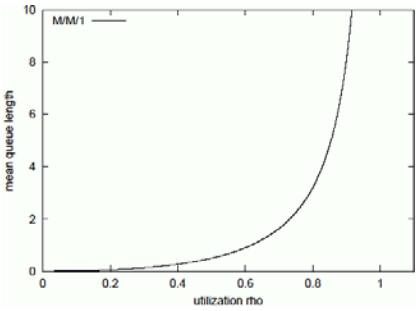


Kendall's Notation

- There are many different queuing systems
- Notation indicates type of arrival and service
  - M – Exponential distribution (memoryless)
  - D – Deterministic distribution
  - G – General distribution
  - …
- Queuing discipline indicates
  - Arrival process
  - Service process
  - Number of servers
- E.g.: M/M/1
  - Simplest case (previous example)

M/M/1 queuing model

- M/M/1 results:
  - Birth-death process with λ and μ
    - $\pi_k=(1-\rho)\rho^k$
    - $\pi_0=1-\rho$
  - Average number of jobs in system
    - $K=\rho/(1-\rho)$
  - Average response time
    - $T=N/\lambda= 1/(\mu\cdot(1-\rho))$
  - Mean queue length
    - $Q=\rho^2/(1-\rho)$
- What are the assumptions?
  - Exponentially distributed interarrival and service times



M/G/1 queuing model

- Service time is not exponentially distributed
  - What does packet transmission time depend on?
    - Packet size
    - Link speed (constant)
- We need different model
  - "Generalized" distribution for service time
- How can we model such a service time?
  - From point of view of arriving job
  - Waiting time depends on
    - Remaining service time of current job (W_0)
    - Sum of mean service times of jobs in queue ($Q \cdot E[X]$)
  - Thus, $W=W_0+Q \cdot E[X]$

M/G/1 queuing model

- Expected service time is independently distributed
  - Use Little's law
    - $W=W_0+Q \cdot E[X] = W_0+\lambda \cdot W \cdot E[X]$
  - With $E[X]=1/\mu$
    - $W=W_0+\rho \cdot W$
  - Solve for W
    - $W=W_0/(1-\rho)$
- What is value of W_0?
  - Depends if server is busy or not
  - $W_0=P[busy] \cdot R+P[not\ busy] \cdot 0$
- How can we determine "mean residual life" R?
  - Result from Kleinrock
    - $R=1/2 \cdot E[X^2]/E[X]=1/2 \cdot E[X](1+c_X^2)$
      - c_X^2, where c is coefficient of variation
      - $c_X=\sigma_X/E[X]$ (normalized standard deviation)

M/G/1 queuing model

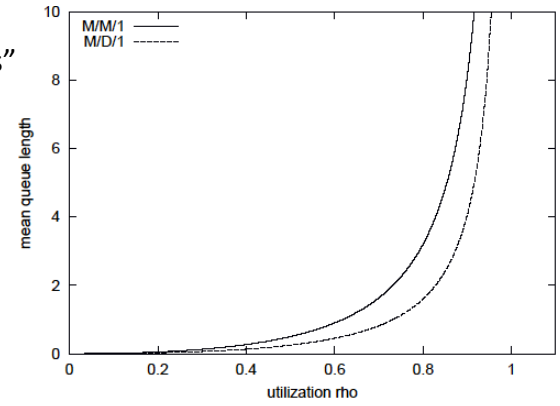
- Total waiting time:
  - $W=W_0/(1-\rho)=\rho/(1-\rho)\cdot 1/2\cdot E[X](1+c_X^2)$
- With Little's law ($Q=\lambda\cdot W$) and $E[X]=1/\mu$:

$$Q=\frac{\rho^2}{(1-\rho)}\cdot\frac{(1+c_X{}^2)}{2}=\frac{\rho^2}{(1-\rho)}\cdot\frac{1}{2}\cdot\frac{E[X^2]}{E[X]^2}$$

  - Pollaczek-Khintchine formula
- Sanity check:
  - Exponential distribution for G
    - $\sigma_X^2=1/\lambda^2$, $E[X]=1/\lambda$, $c_X=\sigma_X/E[X]=1$
    - $Q=\rho^2/(1-\rho)$

M/D/1 queuing model

- Deterministic service time
- Examples
  - Service of "requests"
    - Web page
    - DNS lookup
  - Memory access
- Coefficient of variation $c_X^2=0$
- Queue length
  - $Q=1/2\cdot\rho^2/(1-\rho)$

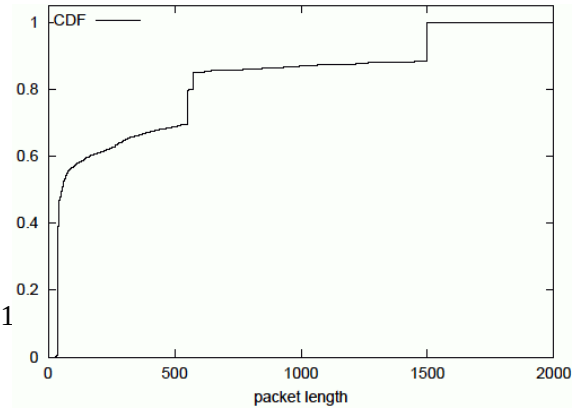


M/G/1 – M/M/1 comparison

- How much do M/G/1 and M/M/1 differ?
  - Assume network traffic
  - M/M/1
    - Service time exponentially distributed
  - M/G/1
    - Service time proportional to packet size
- Queue length
  - M/G/1 queue shorter if $\frac{\rho^2}{(1-\rho)} \cdot \frac{(1+c_X{}^2)}{2} < \frac{\rho^2}{(1-\rho)}$
  - Need c_X^2 for packets
- What is the distribution of packet lengths?

Packet length distribution

- From NLANR:
  - E[X]=354
  - $E[X^2]$=357355
  - σ_X=598
  - c_X=1.687
  - c_X^2=2.844
- Thus
  $\frac{(1+c_X{}^2)}{2} = \frac{(1+2.844)}{2} > 1$
- M/M/1 is too optimistic



Queuing theory summary

- Markov chain as model for stochastic process
  - General solutions for discrete time and continuous time
  - Steady-state probability distribution
- Birth-death process
  - Special case of Markov chain
  - Model for queue in network
  - Closed form solution for delay based on load
- Kendall's notation
  - M/M/1 as simplest queuing model