ECE 671 – Lecture 16

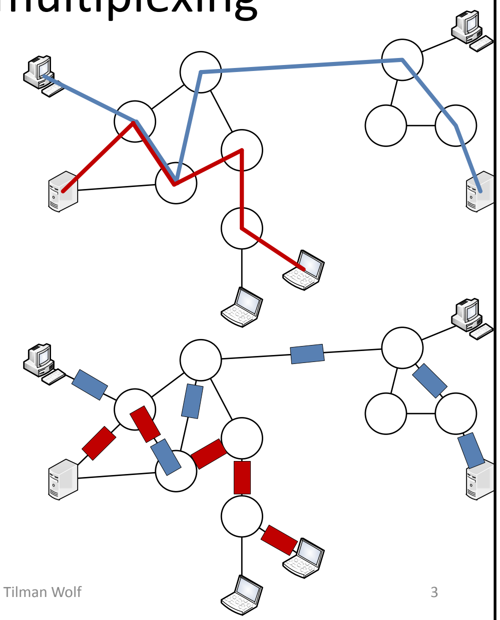
Queuing Theory
Basics

Queuing theory

- Last few lectures:
  - Perspective of single node
  - Focus on input port (i.e., protocol processing)
- Next lectures:
  - Perspective of traffic crossing node
  - Focus on output port (i.e., queuing)
- Queuing theory provides theoretical foundations of Internet

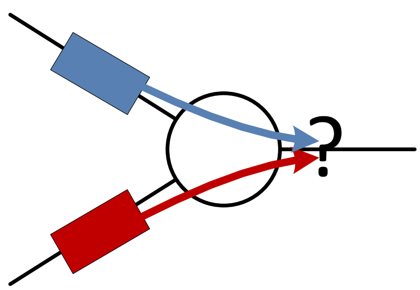
Statistical multiplexing

- Circuit switching
  - Dedicated end-to-end connection
  - Resources are reserved along path
  - Guaranteed constant data rate
  - Achieved trough multiplexing (TDM, FDM)
- Packet switching
  - Packets are unit of transmission
  - “Best effort” and no guarantees
  - Switches perform “store-and-forward”
  - Statistical multiplexing can create contention



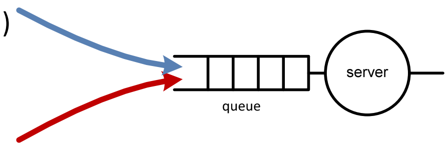
Statistical multiplexing

- What happens here?



Simple queuing example

- Queuing systems are everywhere
  - Line in bookstore (or Blue Wall)
  - Traffic light
  - Your homework assignments
- Key features
  - "Server" has finite capacity (needs time to process)
    - In network terminology, the server is the link
  - Demand for service ("job" arrival) is unpredictable
    - The jobs are the packets
- Questions
  - How long does a job need to wait before being serviced?
  - How many jobs are in the queue?
  - How high is the utilization of the server?

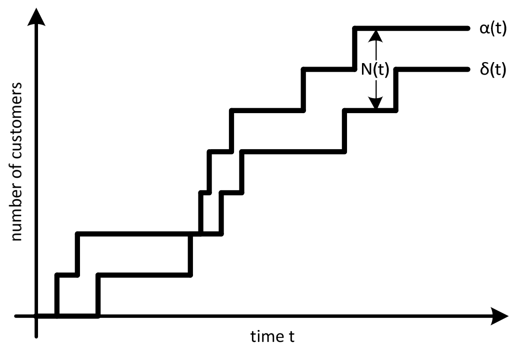


Notation in queuing systems

- Notation introduced by Kleinrock
  - C_n is n^{th} customer entering system
    - τ_n is arrival time for C_n
    - t_n is interarrival time ($t_n=\tau_n-\tau_{n-1}$)
    - x_n is service time for C_n
    - w_n is waiting time for C_n
    - s_n is system time (waiting plus queuing) for C_N ($s_n=w_n+x_n$)
  - N(t) is number of customers in system at time t
  - U(t) is amount of unfinished work in system at time t
  - λ is average arrival rate
    - $E[t_n]=1/\lambda$
  - μ is average service rate
    - $E[x_n]=1/\mu$

Basic queuing behavior

- $\alpha(t)$ is number of arrivals in (0,t)
- $\delta(t)$ is number of departures in (0,t)
- Number of customers in system is
  - $N(t)=\alpha(t)-\delta(t)$
- Average system time is
  - Area between $\alpha(t)$ and $\delta(t)$, denoted by $\gamma(t)$
  - $T_t=\gamma(t)/\alpha(t)$



Little's law

- Average arrival rate
  - $\lambda_t=\alpha(t)/t$
- Average system time
  - $T_t=\gamma(t)/\alpha(t)$
- Average number of customers
  - $\overline{N}_t=\gamma(t)/t$
- Substitute $\gamma(t)$ and $\alpha(t)$
  - $\overline{N}_t=\lambda_t T_t$
- For $t\rightarrow\infty$:
  - $\overline{N}=\lambda T$ (Little's law)
- Average number of customers in queuing system is average arrival rate times average system time.

Related results

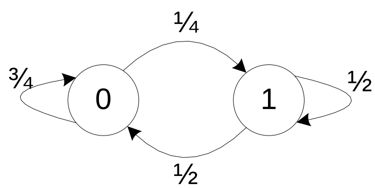
- Average number of customers in queue
  - $\overline{N}_q=\lambda W$
- Relation between waiting and service time
  - $T=\overline{x}+W$
- Utilization ρ
  - $\rho=\lambda/\mu=\lambda\overline{x}$
  - System only stable if $\rho<1$ (why not $\rho=1$?)
  - Let p_0 be probability that server idle: $\rho=1-p_0$
- So far:
  - Not specific to particular type of queue
  - No quantitative results

Modeling of queuing systems

- Any queuing system can be modeled as a "stochastic process"
  - Family of random variables X
    - X(t) is indexed by time parameter $t\in T$
    - $X(t)\in S$, where S is "state space"
  - If S is discrete, then stochastic is a "chain"
- Each state reflects state of queuing system
  - Probabilities indicate what states are more likely
- Markov chains
  - Probability for any state **only** depends on previous state
  - History of Markov chain is summarized in current state

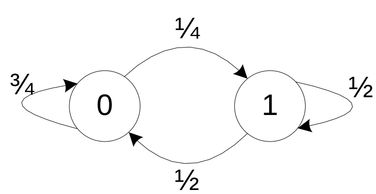
Discrete Time Markov Chains

- DTMC is defined by
  - X_n is random variable indicating state in step n
  - p_{ij} are transition probabilities between states
    - Probability depends on current state only
- Example:
  - State space S={0,1}
  - Transition probabilities P
    - S x S matrix
    - p_{00}=0.75, p_{01}=0.25
    - P_{10}=0.5, p_{11}=0.5
  - Probability to be in state 0 at step n
    - $P[X_n=0] = 0.75 \cdot P[X_{n-1}=0]+0.5 \cdot P[X_{n-1}=1]$



Stationary probability vector

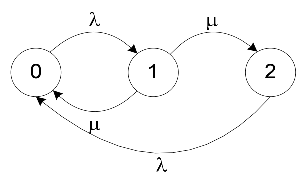
- What is the probability of being in a particular state?
  - If Markov chain "runs long enough", initial state irrelevant
- Define π_i as stationary probability of being in state i
- π_i is independent of time
  - In matrix form: $\pi = \pi P$
- Stationary probability can be solved as set of linear equations:
  - $\pi_0 = 0.75 \cdot \pi_0 + 0.5 \cdot \pi_1$
  - $\pi_1 = 0.25 \cdot \pi_0 + 0.5 \cdot \pi_1$
  - Additional constraint: $\Sigma \pi_i=1$
- Solution: $\pi_0=2/3$, $\pi_1=1/3$



Continuous Time Markov Chains

- Transition between state may happen at any time
- How should probabilities be represented?
  - Probability for infinitesimally small time steps
  - "Transition rate" is suitable description
- "Infinitesimal generator matrix" Q defines rates
  - $q_{ij}(t)=\lim_{\Delta t\to 0}[p_{ij}(t,t+\Delta t)/\Delta t]$ (for $i\neq j$)
  - $q_{ii}(t)=-\Sigma_{j,j\neq i}q_{ij}$
- Example:

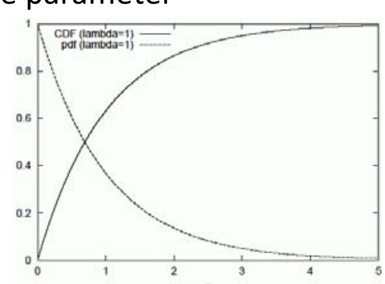
$$Q=\begin{pmatrix}-\lambda & \lambda & 0\\ \mu & -2\mu & \mu\\ \lambda & 0 & -\lambda\end{pmatrix}$$



- Time in a state is memoryless
  - Exponential distribution is memoryless

Exponential distribution

- Exponential distribution has one parameter
  - λ if arrival rate
  - μ if service rate
- Mean: $\overline{X}=1/\lambda$
- CDF: $F_X(r) = 1-e^{-r/\overline{X}}=1-e^{-\lambda r}$
- pdf: $f_X(r) = \lambda e^{-\lambda r}$
- Variance: $var(X) = 1/\lambda^2$



- Convenient properties:
  - Number of arrivals in interval t is Poisson distributed
    - Poisson parameter $\alpha=\lambda t$ and $P[X=k]=\alpha^k\cdot e^{-\alpha}/k!$
  - Rates are additive
    - Combination of two exp. dist. with λ_1 and λ_2 has $\lambda=\lambda_1+\lambda_2$

Steady-state probability vector

- By definition rate of leaving state is rate of staying
  - $q_{ii}(t) = -\Sigma_{j, j \neq i} q_{ij}$
- Steady state probability vector π
  - In steady state, $\pi Q = 0$ or $\Sigma_{i \in S} q_{ij} \pi_i = 0$
    - Change in probability vector is $d\pi j(t)/dt = \Sigma_{i \in S} q_{ij} \pi_i(t)$
    - If steady state, then $\lim_{t \to \infty}[d\pi(t)/dt] = 0$
  - Additional constraint: $\Sigma \pi_i = 1$
- Solution to example:
  - $-\lambda\pi_0 + \mu\pi_1 - \lambda\pi_2 = 0$
  - $\lambda\pi_0 - 2\mu\pi_1 = 0$
  - $\mu\pi_1 - \lambda\pi_2 = 0$
  - Thus, $\pi_1 = \lambda/\mu\pi_2$ and $\pi_0 = 2\pi_2$. With constraint, we get
    - $\pi_0 = 2/(3+\lambda/\mu)$
    - $\pi_1 = \lambda/\mu/(3+\lambda/\mu) = \lambda/(3\mu+\lambda)$
    - $\pi_2 = 1/(3+\lambda/\mu)$

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -2\mu & \mu \\ \lambda & 0 & -\lambda \end{pmatrix}$$