Example: Refraction of light

Assumptions: \(\frac{\partial n(x,z)}{\partial z} = \text{const.}, \quad \alpha_0 = 0, \quad T(h_0) = T_0,\) \(\quad (1)\)

\(\frac{\partial T}{\partial z} = \gamma = \text{const.}, \quad \rho(h_0) = \rho_0\) \(\quad (1)\)

Laws of refraction: \(\frac{\partial x}{\partial x} = -\frac{\partial n}{\partial z}\) \(\quad (2)\)

\[\Rightarrow \quad \alpha(x) = \alpha_0 + \int_{0}^{x} \left( -\frac{\partial n}{\partial z} \right) dx' = \alpha_0 - \frac{\partial n}{\partial z} x\] \(\quad (3)\)

\[\Rightarrow \quad h(x) = h_0 - \int_{0}^{x} \alpha(x') dx' = h_0 - \alpha_0 x + \frac{1}{2} \frac{\partial n}{\partial z} x^2\] \(\quad (4)\)

\[\Rightarrow \quad \alpha(x) = \alpha_0 - \frac{\partial n}{\partial z} x, \quad h(x) = h_0 + \frac{1}{2} \frac{\partial n}{\partial z} x^2\] \(\quad (5)\)
Example (could d)

Optical refractive index for clear air:
\[ n = 1 + \alpha_0 (1 + b \lambda^2) \frac{\rho}{T} \]  
where \( \rho = \text{pressure} \), \( T = \text{temperature} \),
\[ \alpha_0 = 7.76 \cdot 10^{-7} \frac{K}{\text{Pa}} \], \( b = 7.52 \cdot 10^{-3} \text{ mm}^2 \)  

Let \( \lambda = 500 \text{ nm} \) = 0.5 \( \text{ mm} \) \( \Rightarrow b \lambda^2 = 0.050 \)

\[ \Rightarrow \alpha_0 (1 + b \lambda^2) =: \alpha = 7.78 \cdot 10^{-7} \frac{K}{\text{Pa}} \]  

and
\[ n = 1 + \alpha \frac{\rho}{T} \]  

Note: \( n = 1.00029 \) for \( \rho = 1000 \text{ Pa} \), \( T = 273 \text{ K} \)  

\[ \Rightarrow \frac{\partial n}{\partial z} = \alpha \frac{\partial}{\partial z} \left( \frac{\rho}{T} \right) = \alpha \left[ -\frac{\rho}{T^2} \frac{\partial T}{\partial z} + \frac{1}{T} \frac{\partial \rho}{\partial z} \right] \]  

Hydrostatic eq.: \[ \frac{\partial \rho}{\partial z} = -S g \rho \frac{1}{T} \frac{\partial T}{\partial z} \]  

gas eq.

\[ \Rightarrow \frac{\partial n}{\partial z} = -\alpha \frac{\rho}{T^2} \left[ g + \frac{\partial T}{\partial z} \right] \]  

Discussion:

* (5) \( \Rightarrow \) if \( \frac{\partial n}{\partial z} < 0 \), then the ray bends downward:
  \( \alpha \) increases with \( x \), \( n \) decreases with \( x \)
Example (cont'd)

Discussion (cont'd):

- \( \frac{\partial n}{\partial z} < 0 \) (downward bending) if \( \gamma + \frac{g}{R_d} > 0 \),
  
  that is, if \( \gamma = \frac{\partial \tau}{\partial z} > -\frac{g}{R_d} = -0.035 \frac{K}{\text{m}} \)

- Let \( \gamma = 0 \), then \( \frac{\partial n}{\partial z} = -\alpha \frac{P}{T_2} \frac{g}{R_d} \)
  
  \[ \frac{\partial n}{\partial z} = -7.78 \times 10^{-7} \frac{\text{K}}{\text{Pa}} \cdot \frac{10^5 \text{Pa}}{(300 \text{ K})^2} \cdot \frac{10^{4.5}}{287 \text{ kg} \cdot \text{K}} \]
  
  \[ = -3 \times 10^{-8} \frac{1}{\text{m}} \]

- Now, let \( L = 1000 \text{ m} \)
  
  \( \Delta \alpha = \alpha(L) - \alpha_0 = +3 \times 10^{-8} \frac{1}{\text{m}} \cdot 10^3 \text{ m} \)
  
  \[ = 3 \times 10^{-5} \text{ rad} = 30 \text{ \mu rad} \]
  
  \( \Delta h = h(L) - h_0 = \frac{1}{2} \cdot (3 \times 10^{-8} \frac{1}{\text{m}})^2 \cdot 10^3 \text{ m} \)
  
  \[ = -1.5 \text{ cm} \]

Angle of arrival at \( x = L = 1 \text{ km} \):

(5), (12) \( \Rightarrow \alpha(L) = \alpha_0 + \alpha \frac{P}{T_2} (\gamma + \frac{g}{R_d}) L \)

Sensitivities of \( \alpha \) with respect to \( P \) and \( \gamma \):

\( \frac{\partial \alpha}{\partial P} = \frac{\alpha}{T_2} (\gamma + \frac{g}{R_d}) L \approx \frac{\alpha}{T_2} \gamma L = 0.86 \frac{\text{\mu rad}}{\text{hPa}} \)
Example (cont'd)

\[ \frac{\partial x}{\partial y} = a \frac{p}{T^2} L = 860 \frac{\mu \text{rad}}{\text{kPa}} = 0.86 \frac{\mu \text{rad}}{\text{m kPa}} \]

That is, \( x \) is quite insensitive to \( p \) (only \( \sim 1 \mu \text{rad} / \text{kPa} \) or less for \( L = 1 \text{ km} \)) but very sensitive to \( y = \frac{\partial T}{\partial z} \) (\( \sim 1 \mu \text{rad} \) per millikelvin/m for \( L = 1 \text{ km} \)).

That is, a horizontally pointing telescope is a very sensitive instrument for measuring path-averaged \( \frac{\partial T}{\partial z} \) fluctuations.

Refraction of radio waves

Note: In contrast to optical waves, radio waves are strongly affected by humidity.

Refractive index for radio waves:

\[ n = 7.76 \times 10^{-7} \frac{K}{\text{Pa}} \frac{p}{T} + 3.73 \times 10^{-3} \frac{K^2}{\text{Pa}} \frac{p_{\text{w}}}{T^2} \]

where

\( p = \text{total air pressure} \), \( p_{\text{w}} = \text{water vapor partial pressure} \)
Note: Layers of high humidity in the lower troposphere, particularly in combination with temperature inversions, lead to maxima in $n(z)$ and therefore to tropospheric ducting.