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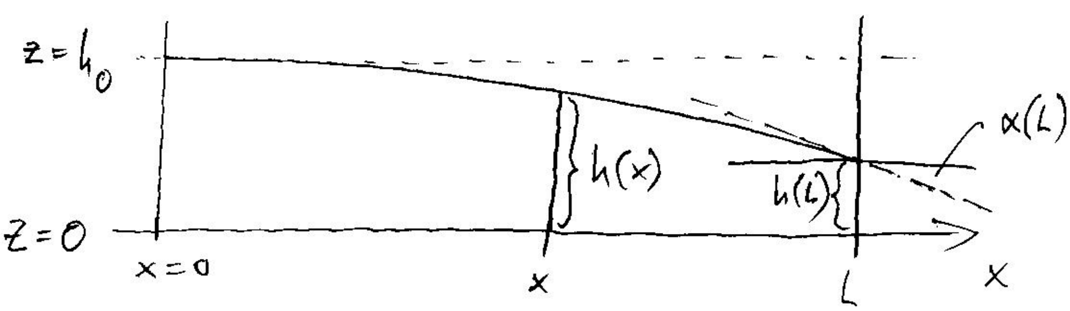
Lecture 22

V. 2. Refraction (cont'd)

Example: Refraction of $\boxed{\text{light}}$

Assumptions: $\frac{\partial n(x,z)}{\partial z} = \text{const.}$, $\alpha_0 = 0$, $T(h_0) = T_0$, (1)

$$\frac{\partial T}{\partial z} = \gamma = \text{const.} \quad , \quad \rho(h_0) = \rho_0 \tag{1}$$



Law of refraction: $\frac{\partial \alpha}{\partial x} = -\frac{\partial n}{\partial z}$ (2)

$$\Rightarrow \quad \alpha(x) = \alpha_0 + \int_{x'=0}^{x} \Big(-\underbrace{\frac{\partial n}{\partial z}}_{=\text{const. here}}\Big) dx' = \alpha_0 - \frac{\partial n}{\partial z} x \tag{3}$$

$$\Rightarrow \quad h(x) = h_0 - \int_{x'=0}^{x} \alpha(x')\,dx' = h_0 - \underbrace{\alpha_0}_{=0 \text{ here}} x + \frac{1}{2}\frac{\partial n}{\partial z} x^2 \tag{4}$$

$$\Rightarrow \quad \boxed{\alpha(x) = \alpha_0 - \frac{\partial n}{\partial z} x \quad , \quad h(x) = h_0 + \frac{1}{2}\frac{\partial n}{\partial z} x^2} \tag{5}$$

Example (cont'd)

Optical refractive index for clear air:

$$n = 1 + a_0 \left(1 + b\lambda^{-2}\right) \frac{p}{T} , \tag{6}$$

where p = pressure, T = temperature,

$$a_0 = 7.76 \cdot 10^{-7} \frac{K}{Pa} , \quad b = 7.52 \cdot 10^{-3} \mu m^2 \tag{7}$$

Let $\lambda = 500\ nm = 0.5\ \mu m \Rightarrow b\lambda^{-2} = 0.030$

$$\Rightarrow a_0\left(1 + b\lambda^{-2}\right) =: a = 7.78 \cdot 10^{-7} \frac{K}{Pa} \tag{8}$$

and $\boxed{n = 1 + a \frac{p}{T}}$ (9)   Note: $n = 1.00029$ for $p = 1000\ hPa$, $T = 273K$

$$\Rightarrow \frac{\partial n}{\partial z} = a \frac{\partial}{\partial z}\left(\frac{p}{T}\right) = a\left[-\frac{p}{T^2}\frac{\partial T}{\partial z} + \frac{1}{T}\frac{\partial p}{\partial z}\right] \tag{10}$$

Hydrostatic eq.: $\frac{\partial p}{\partial z} = -\rho g \underset{\text{gas eq.}}{=} -\frac{p}{R_d T} g$ (11)

$$\Rightarrow \boxed{\frac{\partial n}{\partial z} = -a\frac{p}{T^2}\left[\gamma + \frac{g}{R_d}\right]} \tag{12}$$

Discussion:

- (5) $\Rightarrow$ If $\frac{\partial n}{\partial z} < 0$, then the ray bends downward: α increases with x, h decreases with x

Example (cont'd)

Discussion (cont'd):

- (12) $\Rightarrow \frac{\partial n}{\partial z} < 0$ (downward bending) if $\gamma + \frac{g}{R_d} > 0$,

  that is, if $\gamma \equiv \frac{\partial T}{\partial z} > -\frac{g}{R_d} = -0.035 \frac{K}{m}$

- Let $\gamma = 0$, then $\frac{\partial n}{\partial z} = -a \frac{p}{T^2} \frac{g}{R_d}$

$$\Rightarrow \frac{\partial n}{\partial z} = -7.78 \cdot 10^{-7} \frac{K}{Pa} \cdot \frac{10^5 \, Pa}{(300\,K)^2} \cdot \frac{10 \frac{m}{s^2}}{287 \frac{J}{kg\,K}}$$

$$= -3 \cdot 10^{-8} \, m^{-1}$$

- Now, let $L = 1000$ m

$$\overset{(5)}{\Rightarrow} \Delta\alpha = \alpha(L) - \alpha_0 = +3 \cdot 10^{-8} \, m^{-1} \cdot 10^3 \, m$$

$$= 3 \cdot 10^{-5} \text{ rad} = \underline{\underline{30 \, \mu rad}},$$

$$\Delta h = h(L) - h_0 = \frac{1}{2} \cdot (-3 \cdot 10^{-8} \, m^{-1}) \cdot (10^3 \, m)^2$$

$$= \underline{\underline{-1.5 \text{ cm}}}$$

Angle of arrival at $x = L = 1$ km:

$$(5), (12) \Rightarrow \alpha(L) = \alpha_0 + a \frac{p}{T^2} \left(\gamma + \frac{g}{R_d}\right) L$$

Sensitivities of α with respect to p and γ:

$$\frac{\partial \alpha}{\partial p} = \frac{a}{T^2}\left(\gamma + \frac{g}{R_d}\right) L \approx \frac{a}{T^2} \gamma L \overset{\gamma = 1 \frac{K}{m}}{=} 0.86 \frac{\mu rad}{hPa}$$

Example (cont'd)

$$\frac{\partial \alpha}{\partial \gamma} = a \frac{p}{T^2} L = 860 \frac{\mu rad}{K/m} = 0.86 \frac{\mu rad}{mK/m}$$

That is, α is quite insensitive to p (only $\sim 1\,\mu rad/kPa$ or less for $L = 1\,km$) but very sensitive to $\gamma \equiv \frac{\partial T}{\partial z}$ ($\sim 1\,\mu rad$ per millikelvin/m for $L = 1\,km$).

That is, a horizontally pointing telescope is a very sensitive instrument for measuring path-averaged $\frac{\partial T}{\partial z}$ fluctuations

Refraction of radio waves

Note: In contrast to optical waves, radio waves are strongly affected by humidity.

Refractive index for radio waves:

$$n = 7.76 \cdot 10^{-7} \frac{K}{Pa} \frac{p}{T} + 3.73 \cdot 10^{-3} \frac{K^2}{Pa} \frac{p_w}{T^2},$$

where

p = total air pressure, p_w = water vapor partial pressure

<u>Note:</u> Layers of high humidity in the lower troposphere, particularly in combination with temperature inversions, lead to maxima in $n(z)$ and therefore to $\boxed{\text{tropospheric ducting}}$.