Steps to estimate power spectra $\phi_{xx}(\omega)$ and $S_{xx}(f)$ from a DT signal $x[n]$

1) get equidistant DT signal $x[n]$, $n = 0, ..., N-1$
2) Apply window of length $L \leq N$: $v[n] = w[n] \cdot x[n]$
3) Calculate the DFTS of $v[n]$: $V[k]$, $k = 0, ..., L-1$
4) Calculate modified periodogram $I(\omega_k)$
5) Repeat steps 2, 3, 4 and calculate the averaged modified periodogram (AMP): $\overline{I(\omega_k)}$
6) Get estimate of power spectra $\phi_{xx}(\omega)$ and $S_{xx}(f)$ from AMP

In detail:

Read section 10.6 in Oppenheim 1999

Step 1: Get $x[n]$, $n = 0, ..., N-1$, sampling period $T_s$

Step 2: Apply a window $w[n]$ of length $L$

$k \cdot x[n] \Rightarrow v[n] = w[n] \cdot x[n]$

Re-index $n$ such that $n = 0, ..., L-1$ within window.

Subtract mean value $\frac{1}{L} \sum_{n=0}^{L-1} v[n]$ from $v[n]$

Step 3: Calculate $V[k] = \frac{1}{L} \sum_{n=0}^{L-1} v[n] e^{-j2\pi \frac{k}{L} n}$, $k = 0, ..., L-1$

(that is, the DFTS of $v[n]$)

by means of an "FFT" algorithm.
Step 4: The **modified periodogram** is defined as

\[ I(\omega) = \frac{1}{LU} \left| V(e^{j\omega}) \right|^2, \]

where

\[ V(e^{j\omega}) = \sum_{n=0}^{L-1} v[n] e^{-j\omega n} \]

is the DTFT of \( v[n] \) and

\[ U = \frac{1}{L} \sum_{n=0}^{L-1} (\omega[n])^2 \]

is the variance loss factor due to windowing.

The periodogram is called "modified" if \( \omega[n] \) is not the rectangular window.

For the rectangular window, \( U = 1 \).

In practice, a **sampled version** of \( I(\omega) \) is calculated from \( V[k] \):

\[ I(\omega_k) = \frac{1}{LU} \left| V[k] \right|^2, \quad \omega_k = k \frac{2\pi}{L}, \quad k = 0, \ldots, L-1 \]

Step 5: It can be shown that the expected modified periodogram, \( \langle I(\omega_k) \rangle \) is a good estimate of the power spectrum \( \phi_{xx}(\Omega_k) \):

\[ \langle I(\omega_k) \rangle \approx \frac{1}{T_0} \phi_{xx}(\Omega_k), \quad \Omega_k = \frac{\omega_k}{T_s}, \quad |\omega_k| < \pi \]

assuming that \( x[n] \) has been sampled without aliasing (that is, before sampling, \( x_c(t) \) has been lowpass-filtered with cutoff frequency \( f_c = \frac{1}{2T_s} \) or \( \Omega_c = \frac{\pi}{T_s} \)).
Step 5 (cont'd) ... and assuming that $L$ is not too small.

$$\phi_{xx}(\Omega_k) = T_s \left< I(\omega_k) \right>, \Omega_k = k \frac{2\pi}{T_s}, k = 0 \ldots L-1$$

In practice, $\left< I(\omega_k) \right>$ is estimated from the averaged modified periodogram $I(\omega_k)$:

$$\left< I(\omega_k) \right> \approx \frac{1}{k} \sum_{\tau=0}^{k-1} I_r(\omega_k)$$

where $I_r(\omega_k)$ is calculated from $x[n] = w_r[n] x[n]$.

Length of $x[n]$ segment from which $I(\omega_k)$ is calculated.

This method is due to WELCH (1970).

Step 6: Often, the one-sided power spectrum $S_{xx}(f)$ is used, which satisfies

$$\sigma_x^2 = \int_0^\infty S_{xx}(f) df.$$ 

Then:

$$S_{xx}(f_k = k \cdot \frac{1}{T_s}) = \frac{2T_s}{LU} \left< |V[k]|^2 \right>, \quad k = 1 \ldots \frac{L}{2}$$