IV. Boundary layer and turbulence (cont'd)

9. Taylor's frozen-turbulence hypothesis (1938)


Taylor's hypothesis: In many cases, the spatial structure (in the streamwise direction) of a turbulent flow can be reconstructed from a time series measured with a point sensor.

\[
\begin{align*}
\text{mean flow, speed } U \\
\text{flow, speed } U \\
x_0 - U t & \quad x_0 \\
\rightarrow & \quad \rightarrow \\
\end{align*}
\]

If the turbulent field \( \Psi(x) \) is "frozen" in the mean flow, then

\[
\Psi(x_0 - U t, t) \approx \Psi(x_0, t + \tau)
\]

\[
\Psi(x - U t, t) \approx \Psi(x, t + \tau)
\]

(1)
Therefore, the streamwise spatial ACF of $\psi$ is
\[ \mathbb{B}_{\parallel}(r) = \mathbb{B}_{\parallel}(Ur) = \mathbb{B}_{tt}(r), \]  (2)
where \[ \mathbb{B}_{tt}(r) = \langle \psi'(t) \psi'(t+r) \rangle \]  (3)

Recall the Wiener - Khinchin theorem:

Temporal ACF: \[ \mathbb{B}_{tt}(r) = \int_{-\infty}^{\infty} e^{j\omega r} \phi(\omega) \, d\omega \]  (4)

1-D spatial ACF: \[ \mathbb{B}_{\parallel}(r) = \int_{-\infty}^{\infty} e^{jk_1r} \Phi_{\parallel}(k_1) \, dk_1 \]  (5)

Now, because of (2), we have
\[ \int_{-\infty}^{\infty} e^{j\omega r} \phi(\omega) \, d\omega = \int_{-\infty}^{\infty} e^{jk_1r} \Phi_{\parallel}(k_1) \, dk_1 \]

\[ r = Ur, \quad k = \frac{2\pi}{A} = \frac{2\pi}{UT} = \frac{\omega}{U} \]

\[ k_1 = \frac{2\pi}{\Lambda} = \frac{2\pi}{UT} = \frac{\omega}{U} \]

\[ \Rightarrow \phi(\omega) = \frac{1}{U} \Phi_{\parallel}(k_1 = \frac{\omega}{U}) \quad \text{(from } \Phi_{\parallel}(k_1) \text{ to } \phi(\omega)) \]

\[ \Rightarrow \Phi_{\parallel}(k_1) = U \phi(\omega = Uk_1) \quad \text{(from } \phi(\omega) \text{ to } \Phi_{\parallel}(k_1)) \]
10. Estimation of $\Phi(\omega)$ from DT series

Note: This subsection deals with discrete-time signal processing ("digital signal processing", DSP), and we follow the notation of Oppenheim, Schafer and Buck (1999), particularly Chapter 10.

Def.: $x(t)$ is a continuous-time (CT) signal.

$x[n]$ is a discrete-time (DT) signal.

Note: DT signals can be thought of as sequences of samples of CT signals:

\[ x[n] = x_c(t = nT_s) \]

where $n = \text{discrete time}, \ n \in \mathbb{Z}$,

$t = \text{continuous time}, \ t \in \mathbb{R}$,

$T_s = \text{sampling period (or sampling time)}, \ T = \text{const.}$

Note: $T_s = \text{const.}$ implies that $x[n]$ is a sequence of equidistant samples.
Recall: There are four different Fourier representations.

**CTFT**

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} \, d\Omega, \]

\[ X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} \, dt. \]

**DTFT**

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega, \]

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}. \]

**CTFS**:

\[ x(t+T) = x(t) \forall t \in \mathbb{R}, \]

\[ \Omega_o = \frac{2\pi}{T}, \quad \Omega_k = k\Omega_o. \]

\[ a_k = \frac{1}{T} \int_{t=t_0}^{t_0+T} x(t) e^{-j\Omega_k t} \, dt. \]

**DTFS**:

\[ x[n+N] = x[n] \forall n \in \mathbb{Z}, \]

\[ \omega_o = \frac{2\pi}{N}, \quad \omega_k = k\Omega_o. \]

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\omega_k n}. \]

\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n}. \]