E. Pressure, force and wind

Reading assignment: Chapter 7 (pp. 93 ff.) of Houghton, The physics of atmospheres

E.1. The material derivative

Two descriptions of fluids:
- in a Lagrangian (streamline-following) coordinate system
- in an Eulerian ("fixed") coordinate system

Consider a scalar $\Phi$ in a velocity field $\mathbf{V} = (u, v, w)$:

Then the total derivative of $\Phi$ is

$$
\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} u + \frac{\partial \Phi}{\partial y} v + \frac{\partial \Phi}{\partial z} w
$$

\[ \Rightarrow \]

"total derivative"

"material derivative"

"Lagrangian derivative"

"local time derivative"

"Eulerian time derivative"
Def.: A scalar $\phi$ is called a **conserved scalar** if \[
\frac{d\phi}{dt} = 0.
\]

E.2 Equations of motion

Now, apply Newton’s 2nd law to a fluid parcel:

\[
\frac{dV}{dt} = \mathbf{f'} - \frac{1}{\rho} \nabla p + F,
\]

Navier-Stokes equation, in an inertial frame of reference

where

$\mathbf{f'}$ = a gravitational force,

$\rho$ = fluid (here: air) density,

$p$ = pressure,

$F$ = frictional force,

$-\frac{1}{\rho} \nabla p$ = pressure gradient force

Now, consider a rotating frame of reference $\Sigma'$ (earth) and a non-rotating, non-accelerating, “inertial” f.o.r. $\Sigma'$, where $\Sigma'$ rotates about $\Sigma$ with the angular velocity vector $\Omega$. 
Consider a vector \( \mathbf{A} \). Then,
\[
\frac{d\mathbf{A}}{dt} = \left( \frac{d\mathbf{A}}{dt} \right)_{\Sigma'} + \Omega \times \mathbf{A} \tag{2}
\]

Now, apply (2) for \( \mathbf{A}' = \frac{d\mathbf{A}}{dt} \)
\[
\frac{d^2\mathbf{A}}{dt^2} = \left( \frac{d^2\mathbf{A}}{dt^2} \right)_{\Sigma'} + 2 \Omega \times \left( \frac{d\mathbf{A}}{dt} \right)_{\Sigma'} + \Omega \times (\Omega \times \mathbf{A}) \tag{3}
\]

Now, let \( \mathbf{A} = \mathbf{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \)

\[
\Rightarrow \quad \frac{d\mathbf{V}}{dt} + 2 \Omega \times \mathbf{V} + \Omega \times (\Omega \times \mathbf{V}) = -\frac{1}{s} \nabla \rho + \mathbf{g}' + \mathbf{F} \tag{4}
\]

Equation of motion in a rotating f.o.r.

Now, define the effective gravity \( \mathbf{g} \) as the sum of "real" gravity \( g' \) and centrifugal force:
\[
\mathbf{g} = \mathbf{g}' - \Omega \times (\Omega \times \mathbf{r}) \tag{5}
\]

(5) in (4)
\[
\Rightarrow \quad \frac{d\mathbf{V}}{dt} = 2 \mathbf{V} \times \Omega - \frac{1}{s} \nabla \rho + \mathbf{g} + \mathbf{F} \tag{6}
\]
Now, introduce meteorological coordinates:

North pole

Equator

\( \phi = \text{latitude} \)

\( x = \text{east} \), \( y = \text{north} \), \( z = \text{up} \)

\( u = \text{eastward (zonal) wind} \),
\( v = \text{northward (meridional) wind} \),
\( w = \text{vertical (upward) wind} \)

Then, for points in the atmosphere:

\[
\frac{dV}{dt} = f V \times k - \frac{1}{8} \nabla p + g + F \tag{7}
\]

where

\( k = (0, 0, 1) \) is the "up vector",

\( f = 2 \Omega \sin \phi = \text{"Coriolis parameter"} \)
E. B approximations of equation of motion (7)

E. B. (a) No motion (hydrostatics)

\[ \frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} = 0 \quad \Rightarrow \quad -\frac{1}{g} \frac{\partial}{\partial z} \rho + \frac{\partial}{\partial z} f = 0, \]

\[ f = \left( \begin{array}{c} 0 \\ -g \end{array} \right) \]

\[ \Rightarrow \quad \frac{\partial}{\partial x} \rho = 0, \quad \frac{\partial}{\partial y} \rho = 0, \quad -\frac{1}{g} \frac{\partial}{\partial z} \rho - f = 0 \]

\[ \Rightarrow \quad -\frac{1}{g} \frac{\partial}{\partial z} \rho = f \]

\[ \Rightarrow \quad \frac{\partial}{\partial z} \rho = -fg \quad \text{(hydrostatic equation)} \]

Recall: \( \rho = SR^2 \) (gas eq.)

\[ \Rightarrow \quad \frac{\partial}{\partial z} \rho = -\frac{p}{RT} g \quad \Rightarrow \quad \frac{\partial}{\partial z} \rho = -\frac{g}{R \frac{1}{T} \rho} = \frac{1}{H} \]

where \( H = \frac{RT}{g} \) is the scale height

Note: \( R = 287 \frac{J}{kg \cdot K}, \quad T = 300 K, \quad g = 10 \frac{m}{s^2} \)

\[ \Rightarrow \quad H = 8.61 \text{ km} \]
Example: Change of air pressure with height

sea level: \( P = 1000 \text{ hPa} \)

\[
\frac{\partial P}{\partial z} = -\frac{P}{\eta} = -\frac{1000 \text{ hPa}}{8.6 \text{ km}} \approx -\frac{1 \text{ hPa}}{8.6 \text{ m}}
\]

Example: Increase of water pressure with depth

\[
\frac{\partial P}{\partial z} = -sg \quad \text{where } s \text{ is constant } = 1000 \frac{\text{kg}}{\text{m}^3}
\]

\[
= -10^3 \frac{\text{kg}}{\text{m}^3} \cdot 10 \frac{\text{m}}{s^2} = -10^4 \frac{\text{Pa}}{\text{m}} = -10^5 \frac{\text{Pa}}{10 \text{ m}}
\]

\[
= -\frac{1 \text{ bar}}{10 \text{ m}}
\]