Solutions to Homework #7

Problem 1

(a) $\int_{-\infty}^{\infty} p_x(x)\,dx = 1$ for all p.d.f.'s

$$\Rightarrow a\int_{-\infty}^{\infty} e^{-\frac{(x-b)^2}{c^2}}\,dx = 1 \Rightarrow a = \frac{1}{\int_{-\infty}^{\infty} e^{-\frac{(x-b)^2}{c^2}}\,dx}$$

Let $y = x-b$, $dy = dx$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{(x-b)^2}{c^2}}\,dx = \int_{-\infty}^{\infty} e^{-\frac{y^2}{c^2}}\,dy = 2\underbrace{\int_0^{\infty} e^{-\frac{1}{c^2}y^2}\,dy}_{=\frac{1}{2}\sqrt{\frac{\pi}{1/c^2}}}$$

$$= \sqrt{\pi}\,c$$

$$\Rightarrow \boxed{a = \frac{1}{\sqrt{\pi}\,c}}$$

(b) $\langle x\rangle = \int_{-\infty}^{\infty} p_x(x)\,x\,dx = \int_{-\infty}^{\infty} a\,e^{-\frac{(x-b)^2}{c^2}}\,x\,dx$

$\left.\begin{array}{l} y=x-b \\ x=y+b \\ dx=dy \end{array}\right\} \rightarrow$

$$= a\int_{-\infty}^{\infty} e^{-\frac{y^2}{c^2}}(y+b)\,dy = \underbrace{a\int_{-\infty}^{\infty} e^{-\frac{y^2}{c^2}}\,y\,dy}_{=0} + \underbrace{ab\int_{-\infty}^{\infty} e^{-\frac{y^2}{c^2}}\,dy}_{=ab\cdot\frac{1}{a}}$$

$$= \boxed{b}$$

$\boxed{\text{Problem 1}}$ (cont'd)

(c) Standard deviation: $\sigma_x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$

That is $\sigma_x^2 = \langle y^2 \rangle$, where $y = x - \langle x \rangle = x - b$ is the fluctuation of x.

$$\rho_Y(y) = a\, e^{-\frac{y^2}{c^2}}$$

$$\Rightarrow \sigma_x^2 = \langle y^2 \rangle = \int_{-\infty}^{\infty} a\, e^{-\frac{y^2}{c^2}} y^2\, dy = 2a \int_0^{\infty} e^{-\frac{y^2}{c^2}} y^2\, dy$$

From integral table: $\displaystyle\int_0^{\infty} e^{-\alpha y^2} y^{2k}\, dy \underset{\substack{\uparrow \\ k \in \mathbb{N},\\ \alpha > 0}}{=} \frac{(2k-1)!!}{2^{k+1}\, \alpha^k} \sqrt{\frac{\pi}{\alpha}}$,

where $n!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot n$ is the $\boxed{\text{double factorial}}$.

Here $k = 1 \Rightarrow (2k-1)!! = 1!! = \underline{\underline{1}}$

$$\Rightarrow \int_0^{\infty} e^{-\alpha y^2} y^2\, dy = \frac{1}{4\alpha} \sqrt{\frac{\pi}{\alpha}}, \quad \alpha = \frac{1}{c^2}, \quad a = \frac{1}{\sqrt{\pi}\, c}$$

$$\Rightarrow \sigma_x^2 = 2 \underbrace{\frac{1}{\sqrt{\pi}\, c}}_{a} \cdot \frac{1}{4 \frac{1}{c^2}} \sqrt{\frac{\pi}{1/c^2}} = \frac{c^2}{2} \Rightarrow \boxed{\sigma_x^2 = \frac{c^2}{2}}$$

$\Rightarrow c^2 = 2\sigma_x^2$ That is, the denominator of the exponent of the Gaussian function is always $\boxed{\text{twice the variance}}$, regardless of $\langle x \rangle$.

$\boxed{\text{Problem 1}}$ (cont'd)

(d) Skewness: $Sk_x = \dfrac{\langle (x-\langle x\rangle)^2 \rangle}{\sigma_x^3} \underset{\substack{y=x-\langle x\rangle,\\ \sigma_y^2=\sigma_x^2}}{=} \dfrac{\langle y^3\rangle}{\sigma_y^3}$

$$\langle y^3 \rangle = \int_{-\infty}^{\infty} p_y(y)\, y^3 dy = a \int_{-\infty}^{\infty} \underbrace{\underbrace{e^{-\frac{y^2}{c^2}}}_{\text{even}}\ \underbrace{y^3}_{\text{odd}}}_{\text{odd}} dy = 0$$

because the integral from $-\infty$ to ∞ over an odd function is always zero.

$$\Rightarrow \boxed{Sk_x = 0}$$

(e) Kurtosis: $K_x = \dfrac{\langle (x-\langle x\rangle)^2\rangle}{\sigma_x^4} = \dfrac{\langle y^4\rangle}{\sigma_y^4}$, $y = x-\langle x\rangle$, $\sigma_y = \sigma_x$

$$\langle y^4 \rangle = \int_{-\infty}^{\infty} p_y(y)\, y^4 dy = \int_{-\infty}^{\infty} a\, e^{-\frac{y^2}{c^2}}\, y^4\, dy$$

$$= 2a \underbrace{\int_0^{\infty} e^{-\alpha y^2}\, y^{2k}\, dy}_{= \frac{(2k-1)!!}{2^{k+1}\alpha^k}\sqrt{\frac{\pi}{\alpha}} = \frac{3}{8\, c^{-4}}\sqrt{\pi}\, c}, \quad \text{where } \alpha = \frac{1}{c^2},\ k=2$$

$$= 2\frac{1}{\sqrt{\pi}\, c} \cdot \frac{3}{8}\sqrt{\pi}\, c^5 = \frac{3}{4}c^4 = \frac{3}{4}(\underbrace{2\sigma_x^2}_{c^2})^2 = 3\sigma_x^4$$

$\boxed{\text{Problem 1}}$ (cont'd)

(e) (cont'd)

$$\Rightarrow K_x = \frac{\langle y^4 \rangle}{\sigma_x^4} = \frac{3\sigma_x^4}{\sigma_x^4} \Rightarrow \boxed{K_x = 3}$$

$\boxed{\text{That is, the kurtosis of a Gaussian p.d.f. is always 3.}}$

Therefore, a given p.d.f. cannot be Gaussian if $K_x \neq 3$.

$\boxed{\text{Problem 2}}$

(a) $\langle q \rangle = \int_{-\infty}^{\infty} p_q(q)\, q\, dq$, so $\langle q \rangle$ is known if $p_q(q)$ is known.

If a is a certain value, then the probability that $q > a$ is

$$P[q>a] = \int_a^{\infty} p_q(q)\, dq$$

Now, let $a = \langle q \rangle$

$$\Rightarrow P[q>\langle q \rangle] = \int_{\left(\int_{-\infty}^{\infty} p_q(q) q\, dq\right)}^{\infty} p_q(q)\, dq$$, which can be evaluated if $p_q(q)$ is given.

$\boxed{\text{Problem 2}}$ (cont'd)

(b) The p.d.f. of q is defined as

$$p_q(\tilde{q}) = \lim_{\substack{\varepsilon\to 0,\\ \varepsilon>0}} \frac{P[\tilde{q} < q \leq \tilde{q}+\varepsilon]}{\varepsilon},$$

where $P[\tilde{q} < q \leq \tilde{q}+\varepsilon]$ is the probability that a sample q of the R.V. q is larger than $\tilde{q}$ and smaller than (or equal to) $\tilde{q}$. Because probabilities are always nonnegative and $\varepsilon > 0$, $p_q(\tilde{q}) \geq 0 \ \forall \tilde{q}$.

(c) Given: $Sk_q = 0$

$$\Rightarrow \langle (\underbrace{q-\langle q\rangle}_{y})^3 \rangle = \langle y^3 \rangle = 0$$

$$\Rightarrow 0 = \int_{-\infty}^{\infty} p_y(y)\, y^3 dy = \underbrace{\int_{-\infty}^{0} p_y(y)\, y^3 dy}_{} + \int_{0}^{\infty} p_y(y)\, y^3 dy$$

$\leftarrow z = -y, \ dy = -dz$

$$= \int_{z=+\infty}^{0} p_y(-z)(-z)^3(-dz) = \int_{\infty}^{0} p_y(-z)\, z^3 dz$$

Rename integration variable: $z \to y$

$$= \int_{\infty}^{0} p_y(-y)\, y^3 dy = -\int_{0}^{\infty} p_y(y)\, y^3 dy$$

Problem 2 (cont'd)

(c) (cont'd)

$$\Rightarrow 0 = -\int_0^\infty p_Y(-y)\, y^3\, dy + \int_0^\infty p_Y(y)\, y^3\, dy$$

$$= \int_0^\infty \left[p_Y(y) - p_Y(-y) \right] y^3\, dy$$

Because $y^3 > 0$ within the integration domain,

$p_Y(y) - p_Y(-y) = 0 \quad \forall\, y \geq 0$

$\Rightarrow p_Y(y) = p_Y(-y) \Rightarrow p_Y(y)$ is even

$\Rightarrow p_q(q)$ is symmetrical about $\langle q \rangle$

$$\Rightarrow \boxed{P[q > \langle q \rangle] = \frac{1}{2}}$$

Problem 3

(a) $P = \int_{T_1 = 5^\circ C}^{T_2 = 10^\circ C} p_T(T)\, dT$ is the probability

that $5^\circ C < T \leq 10^\circ C$

Problem 3 (cont'd)

(b) $$P = \int_{T_1 = -40°C}^{T_2 = +40°C} \rho_T(T)\, dT \approx 1$$

because $T < -40°C$ and $T > +40°C$ are extremely rare events (at least since the end of the last ice age) in MA.

(c) $$P[T \text{ is exactly } 20°C] = \lim_{\varepsilon \to 0} \int_{20°C - \frac{\varepsilon}{2}}^{20°C + \frac{\varepsilon}{2}} \rho_T(T)\, dT = 0$$

because $\rho_T(T) < \infty$ if T is a continuous R.V.