Solutions to HW # 6

Problem 1

(a) $$R_{N0} - G_0 = H_0 + \lambda E_0 = 500 \frac{W}{m^2}$$

Bowen ratio: $\mathcal{B} = \frac{H_0}{\lambda E_0}$

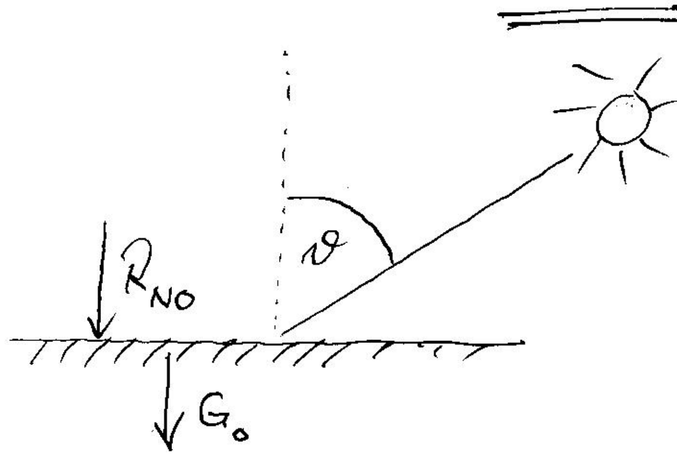
$\mathcal{B} = 3 \Rightarrow H_0 = 3 \lambda E_0 \Rightarrow H_0 \gg \lambda E_0 \Rightarrow$ dry conditions

$$\Rightarrow 500 \frac{W}{m^2} = 3 \lambda E_0 + \lambda E_0 = 4 \lambda E_0$$

$$\Rightarrow \lambda E_0 = \frac{1}{4} \cdot 500 \frac{W}{m^2} = \underline{\underline{125 \frac{W}{m^2}}}$$

$$\Rightarrow H_0 = \mathcal{B} \lambda E_0 = \underline{\underline{375 \frac{W}{m^2}}}$$

(b)



$\vartheta = 60°$

Surface albedo: $\alpha_s = 0.8$

Net surface radiative flux:

$$R_{N0} = R_{S0} - \alpha_s R_{S0} + \varepsilon_s R_{L0}^{d} - R_{L0}^{u}$$

$\boxed{\text{Problem 1}}$ (cont'd)

(b) (cont'd)

Idealized conditions:

$$R_{S0} - \alpha_S R_{S0} = (1-\alpha_S)\, S_0 \cdot \cos \vartheta = 0.2 \cdot 1366 \frac{W}{m^2} \cdot \underbrace{\cos 60^\circ}_{=\frac{1}{2}}$$

$$= 136.6 \frac{W}{m^2}$$

$$R_{L0}^{d} = 0 \quad , \quad R_{L0}^{u} = \varepsilon_S\, \sigma\, T_S^{\,4}$$

$$= 1 \cdot 5.67 \cdot 10^{-8} \frac{W}{m^2 K^4} \cdot (273\ K)^4$$

$$= 314.9 \frac{W}{m^2}$$

~ max. possible T for fresh snow

$G_0 = 0$ (let's assume fresh snow is an ideal isolator)

$$\Rightarrow R_{N0} - G_0 = 136.6 \frac{W}{m^2} - \sigma T_s^{\,4}$$

Obviously, the available energy $R_{N0} - G_0$ decreases with increasing T_s and is negative unless it is very cold. The sign changes at $T_s = \left(\frac{136.6\ W/m^2}{\sigma}\right)^{1/4} = 221.5\ K = \underline{\underline{-52^\circ C}}$

That is, $R_{N0} - G_0$ reaches a maximum when T_s reaches its minimum.

$\boxed{\text{Problem 1}}$ (cont'd)

(c) Same as (b), except $R_{NO} - G_0$ is increased by $\varepsilon_S R^d_{LO} = 1 \cdot 5.67 \cdot 10^{-8} \frac{W}{m^2K^4} \cdot [(273-50)K]^4$

$$\overset{!}{=} 140 \frac{W}{m^2}$$

$$\Rightarrow R_{NO} - G_0 = \underbrace{137 \frac{W}{m^2} + 140 \frac{W}{m^2}}_{277 \frac{W}{m^2}} - \sigma T_S^4$$

In this case, the equilibrium surface temperature would be $T_S = -9°C$.

That is, if there is (fresh) snow on the ground, the available energy depends very sensitively on R^d_{LO} from clouds, even very high (and cold) clouds.

$\boxed{\text{Problem 2}}$

Typo 1: Eq. (7.4) should read: $\frac{DV}{Dt} \overset{\downarrow}{=} 2\underline{V} \otimes \underline{\Omega} - \frac{1}{\varrho} \ldots$, not $\frac{DV}{Dt} \overset{\downarrow}{+} 2\underline{V} \otimes \underline{\Omega} \ldots$

Typo 2: In Eq. (7.6), the first term in the parentheses in front of $\underline{j}$ should be $\frac{Dv}{Dt}$, not $\frac{Du}{Dt}$.

Problem 3

Critical Ri is equal to $\frac{1}{4}$ $\Rightarrow$ $\frac{1}{4} = \frac{N^2}{S_c^2}$

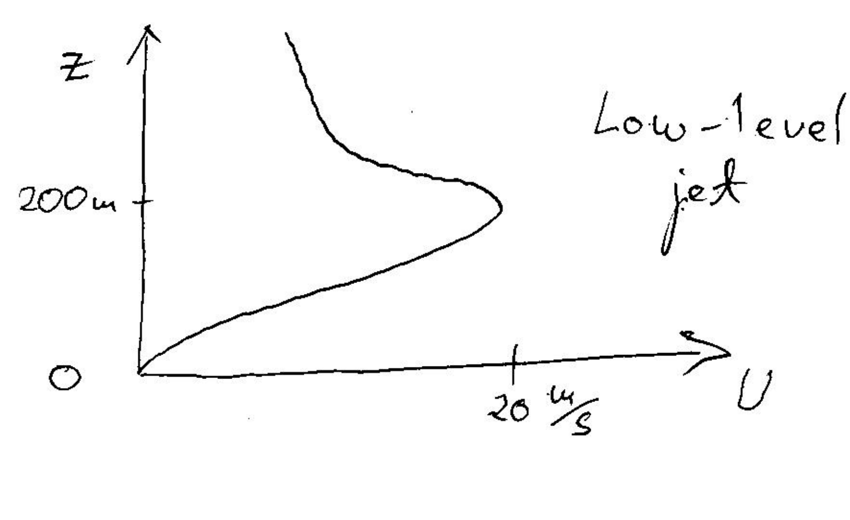
$\Rightarrow |S_c| = 2N$ is the critical shear.

$$\Rightarrow |S_c| = 2\sqrt{\frac{g}{\theta}\frac{\partial \theta}{\partial z}} = 2\sqrt{\frac{g}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right)}$$

$$= 2\sqrt{\frac{10\,\frac{m}{s^2}}{300\,K}\left(10^{-2}\frac{K}{m} + 10^{-2}\frac{K}{m}\right)}$$

$$= 0.052\,\frac{1}{s} = 0.052\,\frac{m/s}{m} = \underline{\underline{5.2\,\frac{m/s}{100\,m}}}$$

Low-level jets with $U = 20\,\frac{m}{s}$ at $z = 200\,m$ are quite common. Because $U = 0$ at the ground (no-slip condition), shear rates of $\frac{20\,m/s}{200\,m} = 0.1\,\frac{1}{s}$ and more are to be expected. Such low-level jets are often turbulent, which is consistent with our result above.



$\boxed{\text{Extra-credit problem}}$

(a) Isothermal atmosphere: $T(z) = T_0 = \text{const.}$

Gas eq.: $p = \varrho R T \Rightarrow \varrho = \frac{p}{RT} = \frac{p}{RT_0}$

Hydrostatic eq.: $dp = -\varrho g \, dz$

$$\Rightarrow dp = -\frac{p}{RT_0} g \, dz$$

$$\Rightarrow \frac{dp}{p} = -\frac{g}{RT_0} dz$$

$$\Rightarrow \int_{p(z=0)=p_0}^{p(z)} \frac{dp}{p} = \int_{z=0}^{z} \left(-\frac{g}{RT_0}\right) dz$$

$$\Rightarrow \ln\left(\frac{p(z)}{p_0}\right) = -\frac{g}{RT_0} z = -\frac{z}{H}, \quad H = \frac{RT_0}{g}$$

$$\Rightarrow \boxed{p(z) = p_0 \, e^{-\frac{z}{H}}}, \quad H = \frac{RT_0}{g} = \text{"scale height"}$$

(b) Polytropic atmosphere: $T(z) = T_0 - \gamma z$,
T_0, γ const.

Gas eq. + hydr. eq. $\Rightarrow dp = -\frac{p \, g}{R \, T(z)} dz$

$$\Rightarrow \frac{dp}{p} = -\frac{g}{R(T_0 - \gamma z)} dz$$

Extra-credit problem (cont'd)

Substitution : $y = T_0 - \gamma z$, $\frac{dy}{dz} = -\gamma$, $dz = -\frac{1}{\gamma} dy$

$$\Rightarrow \frac{dp}{p} = -\frac{g}{Ry}\left(-\frac{1}{\gamma} dy\right) = \frac{g}{R\gamma}\frac{dy}{y}$$

$$\Rightarrow \int_{p_0}^{p(z)} \frac{dp}{p} = \int_{y_0}^{y(z)} \frac{g}{R\gamma}\frac{dy}{y}$$

$$\Rightarrow \ln\frac{p(z)}{p_0} = \frac{g}{R\gamma}\ln\frac{y(z)}{y_0} = \frac{g}{R\gamma}\ln\frac{T_0-\gamma z}{T_0}$$

$$\Rightarrow \frac{p(z)}{p_0} = e^{\frac{g}{R\gamma}\ln\frac{T_0-\gamma z}{T_0}} = \left(e^{\ln\frac{T_0-\gamma z}{T_0}}\right)^{g/R\gamma}$$

$$\Rightarrow \boxed{p(z) = p_0\left(\frac{T_0-\gamma z}{T_0}\right)^{g/R\gamma}}$$

(c) Recall : $e = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]$

Here : $\frac{T_0-\gamma z}{T_0} = 1 - \frac{\gamma z}{T_0}$, so let $\frac{1}{x} = -\frac{\gamma z}{T_0}$

$\boxed{\text{Extra-credit problem}}$ (cont'd)

$$\leadsto \qquad x = -\frac{T_0}{\gamma z} \quad , \quad \frac{1}{\gamma} = -\frac{z}{T_0} x$$

$$\lim_{\gamma \to 0} \left(1 - \frac{\gamma z}{T_0}\right)^{g/R\gamma}$$

$$= \left(1 + \frac{1}{x}\right)^{\frac{g}{R}\left(-\frac{z}{T_0} x\right)}$$

$$= \left[\left(1 + \frac{1}{x}\right)^{x}\right]^{-\frac{g}{RT_0} z}$$

$$\Rightarrow \lim_{\gamma \to 0} p(z) = \lim_{x \to \infty} \left(p_0 \left[\left(1 + \frac{1}{x}\right)^{x}\right]\right)^{-\frac{g}{RT_0} z}$$

$$= p_0 \cdot \left[\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}\right]^{-\frac{g}{RT_0} z} = p_0 \, e^{-\frac{g}{RT_0} z} \; /$$

q.e.d.