Solutions to HW #2

Problem 1:

Ideal delay system: $y(t) = x(t - T_d)$

(a) $y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$ for LTI systems

from input/output relationship

$$= x(t - T_d) = \int_{-\infty}^{\infty} \delta(\tau - T_d)\, x(t-\tau)\, d\tau$$

$$\Longrightarrow h(\tau) = \delta(\tau - T_d) \Rightarrow \boxed{h(t) = \delta(t - T_d)}$$

(b) $H(j\Omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j\Omega t}\, dt$

$$= \int_{-\infty}^{\infty} \delta(t - T_d)\, e^{-j\Omega t}\, dt = e^{-j\Omega T_d}$$

$$\Longrightarrow \boxed{H(j\Omega) = e^{-j\Omega T_d}}$$

(c) $x(t) = A \cdot \cos(\Omega_0 t + \varphi)$,

$y(t) = A \cdot \cos(\Omega_0 (t - T_d) + \varphi)$,

where $\Omega_0 = \frac{2\pi}{T}$, A = amplitude, and φ = phase at $t = 0$

Measurement error: $e(t) = y(t) - x(t)$

Error variance (mean square): $\sigma^2 = \frac{1}{T}\int_0^T e^2(t)\,dt$

Recall: $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\Rightarrow e(t) = A\left[\cos(\underbrace{\Omega_0 t + \varphi}_{\alpha} - \underbrace{\Omega_0 T_d}_{\beta}) - \cos(\underbrace{\Omega_0 t + \varphi}_{\alpha})\right]$$

Now,

$$\left[\cos(\alpha-\beta) - \cos\alpha\right]^2$$

$$= \left[\cos\alpha\cos\beta + \sin\alpha\sin\beta - \cos\alpha\right]^2$$

$$= \left[\cos\alpha(\cos\beta - 1) + \sin\alpha\sin\beta\right]^2$$

$$= \cos^2\alpha\,\underbrace{(\cos\beta-1)^2}_{a} + 2\sin\alpha\cos\alpha\,\underbrace{(\cos\beta-1)\sin\beta}_{b} + \sin^2\alpha\,\underbrace{\sin^2\beta}_{c}$$

$$= a\cos^2\alpha + 2b\sin\alpha\cos\alpha + c\sin^2\alpha$$

Now, find the mean square of e by integrating α from 0 to 2π and dividing by 2π:

$$\langle e^2\rangle = A^2\left[\underbrace{\frac{a}{2\pi}\int_0^{2\pi}\sin^2\alpha\,d\alpha}_{a/2} + \underbrace{\frac{2b}{2\pi}\int_0^{2\pi}\sin\alpha\cos\alpha\,d\alpha}_{0} + \underbrace{\frac{c}{2\pi}\int_0^{2\pi}\sin^2\alpha\,d\alpha}_{=\frac{c}{2}}\right]$$

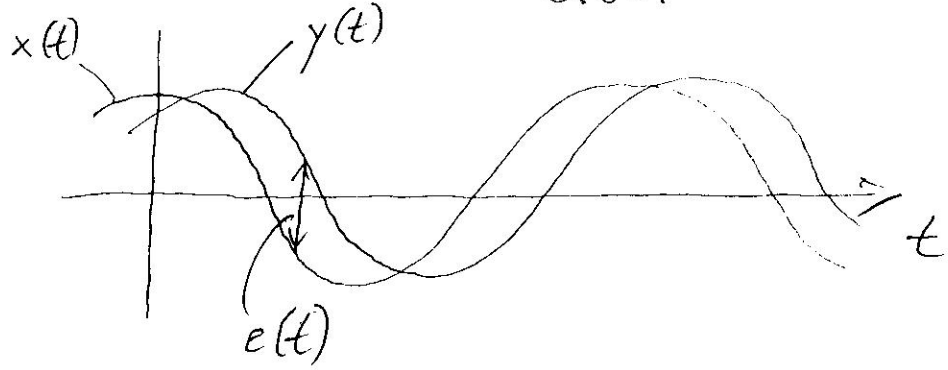
$$= \frac{A^2}{2}\left[(\cos\beta-1)^2 + \sin^2\beta\right]$$

$$\Rightarrow \langle e^2 \rangle = \frac{A^2}{2}\left[\cos^2\beta - 2\cos\beta + 1 + \sin^2\beta\right]$$
$$= \frac{A^2}{2}\left[2 - 2\cos\beta\right]$$
$$= A^2\left(1 - \cos\beta\right)$$

$$\Rightarrow \sigma_e = \sqrt{\langle e^2 \rangle} = A\sqrt{1-\cos\beta} = \underline{\underline{A\sqrt{1-\cos\left(\frac{2\pi}{T}T_d\right)}}}$$

<u>Here:</u> $A = 1\,V$, $T = 10\,\mu s$, $T_d = 1\,\mu s$

$$\Rightarrow \sigma_e = 1V \cdot \sqrt{1 - \underbrace{\cos\left(\frac{2\pi}{10\,\mu s}\cdot 1\,\mu s\right)}_{0.809}} = \underline{\underline{0.437\ V}}$$



(d) From (c), we know:

$$e^2(t) = A^2\left[a\cos^2\alpha + 2b\sin\alpha\cos\alpha + c\sin^2\alpha\right],$$

where $a = (\cos\beta - 1)^2$, $b = (\cos\beta - 1)\sin\beta$, $c = \sin^2\beta$,

$$\beta = \frac{2\pi}{T}T_d, \quad \alpha = \Omega_0 t + \varphi$$

Now, we let T_d be a random variable (R.V.) with a Gaussian p.d.f.:

$$f_{T_d}(T_d) = \frac{1}{\sqrt{2\pi}\,\sigma_{T_d}} e^{-\frac{T_d^2}{2\sigma_{T_d}^2}}$$

(Here, for the sake of simplicity, we let $\langle T_d \rangle = 0$. That is, we assume T_d to be a zero-mean R.V.)

Because $a = a(T_d)$, $b = b(T_d)$, $c = c(T_d)$, the three variables a, b, c are also R.V.'s

$$\Rightarrow \sigma_e^2 = \langle e^2(t) \rangle$$

$$= A^2 \left[\langle a \rangle \cos^2\alpha + 2\langle b \rangle \sin\alpha \cos\alpha + \langle c \rangle \sin^2\alpha \right]$$

Note that at this point σ_e^2 is a function of time!

Now, evaluate the three expected values:

$$\langle a \rangle = \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_{T_d}} e^{-\frac{T_d^2}{2\sigma_{T_d}^2}}}_{f_{T_d}(T_d)} \cdot \underbrace{\left(\cos\frac{2\pi}{T} T_d - 1\right)^2}_{a} dT_d \,,$$

$$\langle b \rangle = \int_{-\infty}^{\infty} f_{T_d}(T_d)\, b \, dT_d \,, \quad \langle c \rangle = \int_{-\infty}^{\infty} f_{T_d}(T_d)\, c \, dT_d \,.$$

These integrals can be evaluated in close form. Then, as the last step, integrate over α from 0 to 2π, as in 1(c).

Problem 2

Voltage error at some time t is the sum of thermal noise $n(t)$ and of digitization noise $q(t)$:

$$e(t) = n(t) + q(t).$$

(a) Precision of voltage measurement for $q(t)=0$:

$$\sigma_V = \sqrt{\langle e_V^2 \rangle} = \sqrt{\langle n^2 \rangle} = \sigma_n = 1\,\mu V$$

Temperature precision:

$$\sigma_T^2 = \langle e_T^2 \rangle = \left\langle \left(\frac{\partial T}{\partial V} e_V \right)^2 \right\rangle = \left(\frac{\partial T}{\partial V} \right)^2 \langle e_V^2 \rangle = \left(\frac{\partial T}{\partial V} \right)^2 \sigma_V^2$$

$$\Rightarrow \sigma_T = \left| \frac{\partial T}{\partial V} \right| \sigma_V \underset{\substack{\uparrow \\ \text{here}}}{=} \frac{1}{40} \frac{K}{\mu m} \cdot 1\,\mu V = \frac{1}{40} K = \underline{\underline{0.025\,K}}$$

(b) $$\sigma_V^2 = \langle e_V^2 \rangle = \langle (n+q)^2 \rangle = \langle n^2 + 2nq + q^2 \rangle$$

$$= \underbrace{\langle n^2 \rangle}_{=\sigma_n^2} + 2 \underbrace{\langle nq \rangle}_{=0 \text{ because } n, q \text{ stat. independent}} + \underbrace{\langle q^2 \rangle}_{=\frac{s^2}{12}}$$

$$= \sigma_n^2 + \frac{1}{12} s^2$$

$$\Rightarrow \sigma_V = \sqrt{\sigma_n^2 + \frac{1}{12} s^2} = \sqrt{(1\,\mu V)^2 + \frac{1}{12}(1\,\mu V)^2} = \sqrt{\frac{13}{12}}\,\mu V$$

$$\Rightarrow \sigma_T = \left| \frac{\partial T}{\partial V} \right| \sigma_V = \frac{1}{40} \frac{K}{\mu V} \cdot \sqrt{\frac{13}{12}}\,\mu V = 0.026\ K$$

Problem 2 (cont'd)

(c) $\sigma_V \leq \frac{11}{10} \sigma_n \quad \Rightarrow \sigma_V^2 \leq \left(\frac{11}{10}\right)^2 \sigma_n^2$

$$\Rightarrow \sigma_n^2 + \frac{1}{12} s^2 \leq \frac{121}{100} \sigma_n^2$$

$$\Rightarrow \frac{1}{12} s^2 \leq \frac{21}{100} \sigma_n^2$$

$$\Rightarrow s^2 \leq \frac{12 \cdot 21}{100} \sigma_n^2 \Rightarrow \underline{\underline{s \leq 1.59\ \sigma_n}}$$

$$\Rightarrow s_T \leq 1.59 \cdot \underbrace{25\,\mu K}_{\sigma_{n,T}} = 39.8\ \mu K$$

Problem 3:

(a) We know: $S_0 = \frac{a}{\vartheta_s^2}$, where $a = \text{const.}$

$$\Rightarrow \frac{dS_0}{d\vartheta_s} = a \cdot (-2) \vartheta_s^{-3} = -\underbrace{\frac{a}{\vartheta_s^2}}_{= S_0} \cdot \frac{2}{\vartheta_s} = -2 \frac{S_0}{\vartheta_s}$$

$$\Rightarrow \frac{\Delta S_0}{\Delta \vartheta_s} \approx -2 \frac{S_0}{\vartheta_s}$$

$$\Rightarrow \frac{\Delta S_0}{S_0} \approx -2 \frac{\Delta \vartheta_s}{\vartheta_s}$$

That is, if ϑ_s <u>dec</u>reases by $p\,\%$, then S_0 <u>inc</u>reases (minus sign!) by $\underline{2p\,\%}$.

Problem 3 (cont'd):

(b) We know: $T_e = b\, S_o^{\frac{1}{4}}$, $b = \text{const.}$

$$\Rightarrow T_e = b \left(\frac{a}{D_s^2}\right)^{\frac{1}{4}} \Rightarrow T_e = c\, D_s^{-\frac{1}{2}}, \quad c = \text{const.}$$

$$\Rightarrow \frac{\Delta T_e}{T_e} \approx -\frac{1}{2} \frac{\Delta D_s}{D_s}$$

$\Rightarrow$ That is, if D_s decreases by $p\%$, then T_s increases by $\frac{p}{2}\%$.

Problem 4:

Let Sun, Earth and Moon be nearly aligned. That is, we have full moon but not a lunar eclipse.