Total Unimodularity

Consider the Linear Programming (LP) problem defined as follows:

$$\min c^T x$$

$$\text{s. to:} \quad \begin{cases} A \cdot x \geq b \\ x \geq 0 \end{cases}$$

Definition (Unimodular Matrix): A square integer matrix A is called *unimodular* if $|det(A)| = 1$. A matrix A is called *totally unimodular* if every square nonsingular submatrix of A is unimodular.

Theorem: Let A be an integer $m \times n$ matrix. If A is *totally unimodular*, then, for all integer m-vectors b, all corners of the convex polytope defined by $A \cdot x = b$, $x \geq 0$, are integer.

The following Lemma provides a convenient means to check for total unimodularity.

Lemma: Let A be an integer matrix with coefficients from {-1, 0, 1}. Assume that the rows of A can be partitioned into two sets R_1 and R_2, such that the following holds:

- If a column has two entries of the same sign, their rows are in different sets.
- If a column has two entries of different signs, their rows are in the same set.

Then, A is *Totally Unimodular*.

A few important remarks:

- Total Unimodularity of the constraint matrix A is a *sufficient* (and, for inequality constraints, also *necessary*) condition for a Linear Program to give integer solutions, provided the constraint vector b is integer.
- A class of graphs, called **Flow Networks**, has a property that their incidence matrix A is *totally unimodular*. Therefore, integral solutions for this class of graphs can be obtained using LP. This is an alternative to specialized graph-theoretic algorithms, such as min-cut/max-flow, min-cost flow, matching, transportation problem, multi-commodity flow, linear assignments, and many others.

Reference:

T. Lengauer, *Combinatorial Algorithms for Integrated Circuit Layout*, Chapter 4.
(Excellent book on combinatorial algorithms, not only for layout; now out of print).

Duality in Linear Programming

Primal variables: $x \geq 0$. Dual variables: $w,\ u,\ v \geq 0$.
$c =$ cost vector, $A =$ constraint matrix, $b =$ constraint vector.

1. **Symmetric forms**

Primal problem	Dual problem
$\min\ c^T x$ s. to: $\begin{cases} Ax \geq b \\ x \geq 0 \end{cases}$	$\max\ w^T b$ s. to: $\begin{cases} A^T w \leq c \\ w \geq 0 \end{cases}$
$\max\ c^T x$ s. to: $\begin{cases} Ax \leq b \\ x \geq 0 \end{cases}$	$\min\ w^T b$ s. to: $\begin{cases} A^T w \geq c \\ w \geq 0 \end{cases}$

2. **Complementary Slackness condition**

At an optimum point: $c^T x = w^T b$, and

$$w^T(Ax - b) = 0$$

$$(c - A^T w)x = 0$$

If x is feasible (optimum) in Primal, w is feasible (optimum) in Dual.

3. **Asymmetric forms**

Primal problem	Dual problem
$\min\ c^T x$ s. to: $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$	$\max\ z^T b$ s. to: $\begin{cases} A^T z \leq c \\ z:\ free \end{cases}$
$\max\ c^T x$ s. to: $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$	$\min\ z^T b$ s. to: $\begin{cases} A^T z \geq c \\ z:\ free \end{cases}$

Proof:
Consider the primal minimization problem. Replace $Ax = b$ by $Ax \geq b$ and $-Ax \geq -b$. Create the dual variables $u,\ v \geq 0$, associated with the first and second constraint set, respectively. Then the cost vector of the dual problem is $[u - v]$, and the dual objective to be maximized is $[u - v]^T b$. The dual constraint set is $A^T u - A^T v = A^T[u - v] \leq c$. Replace $[u - v]$ by z, a free variable, to obtain the required form. *QED.*