Using Rational Approximations For Evaluating The Reliability of Highly Reliable Systems

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Abstract

This paper discusses a new approach for accurately evaluating the reliability of a complex, highly reliable system for which neither analysis nor brute-force simulation are feasible. We propose to calculate the reliability at feasible parameter values, and then use Rational Interpolation to evaluate the desired reliability. We present a detailed case-study to demonstrate the usefulness of this method.

1. Introduction

In this paper, we outline a new approach for evaluating the reliability of complex, highly reliable systems. Increasingly, applications have arisen in which the specified availability or reliability over a given period of operation is of the order of 99.999% or greater. Conventional approaches to evaluating the reliability of such systems have severe limitations. Analytical techniques based on Combinatorics or on Markov processes theory are infeasible in complicated systems owing to the explosive increase in the number of distinct states that have to be considered. Conventional simulation techniques often take a prohibitively long time to execute, due to the very small failure probability.

There are several approaches mentioned in the literature for the calculation of probabilities of rare events, most notably - Importance Sampling. All of these approaches have their drawbacks. We focus here on one of the more promising approaches, Rational Interpolation (RI) [5].

The basis for this technique is as follows. Simulations of highly reliable systems take a long time to execute because failures happen so rarely that gathering sufficient failure statistics is extremely slow. By contrast, we can quickly obtain simulation results of adequate accuracy when simulating under parameters that cause failures to occur more often. The idea is to obtain statistics under the assumption that the failure rates of individual blocks are high. Then, we construct a closed-form rational interpolant to the obtained points. A rational interpolant has the form

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m} \tag{1}$$

The RI in (1) is called an (n, m) RI, x is a parameter such as the component failure rate or the system's operating time, and the constants a_i, b_i are determined by the higher-failure-rate simulation results. Since without loss of generality, we can select $b_0 = 1$, $k = n+m+1$ input points $f(x_1), ..., f(x_k)$ are needed to calculate the coefficients for an (n, m) RI. The points $(x_1, ..., x_k)$ are selected in the region where failure rates are high, k equations of the form (1) are solved for a_i, b_i [4], and then the reliability can be predicted for very low failure rates. Sometimes, based on the type of the function $f(x)$, a pre-transformation is performed on $f(x_1), ..., f(x_k)$ and (1) is solved for the transformed values. The purpose of the pre-transformation is to obtain a functional relationship that can be approximated by a rational approximation at a higher precision.

The RI method was introduced to performance analysis in [5, 9], in which a series of results of Stahl were used [3]. RI has been successfully applied to the analysis of some discrete-event systems [5, 6, 9], the computation of cell loss probability in ATM multiplexers [10], and some other performance analysis problems in computer and queueing systems [7, 8].

This paper demonstrates the use of the Rational Interpolation technique for calculating the accurate reliability of a highly-reliable system. We next outline the motivation behind our model and explain how we suggest to tackle it. We then present some initial numerical results of experiments which we have performed, and outline future directions we intend to pursue.

2. Motivation

There are many ways of defining reliability [11]. Traditional reliability involves a system which can be in one of two unambiguous states: *up* or *down*, and is composed of subsystems which are themselves in one of these states. Examples include series-parallel systems [1] or more general interconnections of modules [13].

More advanced reliability problems arise if we have to take into account the possibility of the system operating in degraded states. In such an event, measures as *performability* or *capacity reliability* can be used, in which the system performance/reliability is expressed as a vector of probabilities.

In the study described here, we considered the evaluation of the traditional static reliability. That is, the system is described as a block diagram of modules, each of which can be in one of two states: up or down. When a module fails, a repair process begins, at the end of which the module is "up" again. The system as a whole is considered to be "up" if certain combinations of modules are up, and down otherwise. We define the reliability of the system at time t as the probability that the system has been up during the whole time interval $[0, t]$.

This paper is meant to be a "proof of concept" for the idea of using Rational Interpolation for the accurate calculation of high reliabilities which would not have been feasible otherwise. To this end, we selected a system which is amenable to an analytic solution and not just to simulation-based results. The behavior of the system selected for demonstrating the RI method is represented by the diagram in Figure 1. We assume that the time to failure and the time

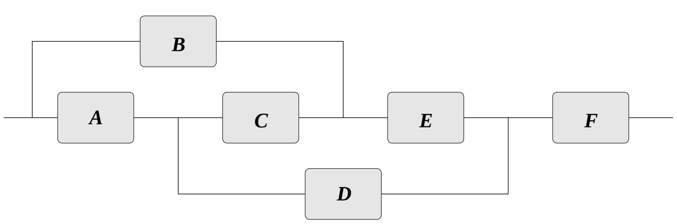

Figure 1. A Non Series/Parallel System.

to repair of module i are exponentially distributed, with parameters λ_i and μ_i, respectively $(i = A, B, ..., F)$.

Very high reliabilities can be the result of low failure rates, high repair rates or small values of the time t. Since we need one parameter that determines the behavior of the system as a whole, we selected as the parameter of the rational interpolant the system time t (the other parameters can be varied accordingly). Also, without loss of generality, we assume that t is integer valued: $t = 0, 1, 2, ...$.

The system is *up* if at least one of the combinations BEF, $ACEF$, or ADF is up, and *down* otherwise. The system reliability at time t, $R(t)$, is defined as the probability that the system has been up during the whole time interval $[0, t]$. Calculating $R(t)$ is not a trivial task. However, the system can be described as a Markov chain with 16 states, denoted by 0,...,15. State 15 denotes the initial state in which all modules are up and state 0 denotes the "system is down" state. Setting state 0 as an absorbing state, the un-reliability of the system at time t is the transition probability $P_{15,0}(t)$, or equivalently, the state probability $P_0(t)$ given that $P_{15}(0) = 1$. The reliability $R(t)$ can be calculated as

$$R(t) = 1 - P_{15,0}(t)$$

Rather than solving a large set of differential equations, the transition probabilities for the Markov chain can be calculated numerically using the powerful uniformization method [2], which provides very accurate results. The availability of an analytical solution allows us to compare the predictions of the RI approach with the exact reliability, as well as with simulation results.

3. The RI Approach

We next give a brief description of the major steps in the rational interpolation approach to reliability calculation.

1. Select a parameter x for the reliability function R.
2. Calculate the asymptotic value of $R(x)$ as $x \to 0$.
3. Obtain interpolation points $R(x_i)$ (for large x_i's) through either analytic calculation or simulation.
4. Perform an appropriate pre-transformation to $R(x)$ suggested by its asymptotic behavior.
5. Construct RI that interpolates the points obtained in Step 3. Generate a sequence of RIs with increasing orders until a specific RI is selected according to some criterion.
6. Transform back the RI result to obtain the reliability values in the range of interest.

As mentioned above, we selected as the main parameter the system time t. Let us denote by $R(t)$ the accurate reliability function, by $R^s(t)$ the reliability as obtained by Monte-Carlo simulation, and by $\hat{R}_{m,n}(t)$ the estimate obtained when using an RI of degree (m, n).

In many cases the reliability of a system can be expressed as a sum of exponential functions, and thus, a logarithmic pre-transformation $log(R(t))$ will make the values easier to interpolate using a rational function.

The main difficulty in implementing the algorithm described above is in step 5. : How do we choose the right RI among the many candidates? We suggest the following procedure. Select three sets of points on the T-axis:

- $x_1, ..., x_k$ are the *input points* - the points for which either the accurate $R(t)$ or the simulated $R^s(t)$ is calculated ($R^s(t)$ will include some noise) and which are used in (1) to produce the coefficients a_i and b_i. Note that for the input points $x_1, ..., x_k$ ($k = m + n + 1$), $\hat{R}_{m,n}(x_i) = R(x_i)$ or $\hat{R}_{m,n}(x_i) = R^s(x_i)$, depending on whether an analytic function or simulation results are used for the reliability calculation.

- $y_1, ..., y_l$ are the *test points* - additional points on the time axis which assist us in choosing the best RI for our purposes. We will calculate the accurate $R(y_i)$ (or simulated $R^s(y_i)$) values, though possibly at a higher cost than calculating $R(x_i)$. In addition, we will calculate (for a given RI of degree (m, n)) the extrapolated values $\hat{R}_{m,n}(y_i)$ using (1), and then obtain the average Y-error as given by

$$\bar{E}_{m,n}(Y) = \frac{\sum_{i=1}^{l} |\hat{R}_{m,n}(y_i) - R(y_i)|}{l} \tag{2}$$

- $z_1, ..., z_s$ are the *target points* - the points for which we are interested in evaluating the reliability. In an actual application, calculating $R(z_i)$ (or even $R^s(z_i)$) is infeasible and only the estimates $\hat{R}_{m,n}(z_i)$ are obtainable. In this paper, both $R(z_i)$ and $\hat{R}_{m,n}(z_i)$ will be calculated, to test the validity of our approach. To this end we define, similarly to (2), the average Z-error for a given RI of degree (m, n),

$$\bar{E}_{m,n}(Z) = \frac{\sum_{i=1}^{s} |\hat{R}_{m,n}(z_i) - R(z_i)|}{s} \tag{3}$$

Note that in (2) and (3), $R(t)$ will in most cases be replaced by $R^s(t)$ since only simulation results will be available.

The approach we study for predicting the reliability of highly reliable systems is as follows:

1. Select target points, test points and input points on the time axis.

2. Get a sequence of rational interpolants of varying (m, n) based on the input points.

3. For each function obtained in step 2., find the average error $\bar{E}_{m,n}(Y)$ over the test points.

4. Select the rational interpolant with the lowest $\bar{E}_{m,n}(Y)$ and use it to predict the reliability for the target points.

4. Numerical Experiments

To demonstrate the effectiveness of the RI method outlined above for reliability estimation, we performed a series of numerical experiments on the system depicted in Figure 1.

The failure/repair rates selected were: $\lambda_i = 0.0001(7 - i)$, $\mu_i = 0.004$ $(i = 1, ..., 6)$. The variable t is the system time, and $R(t)$ is the system reliability at time t $(t = 0, 1, 2, ...)$. Clearly, $R(0) = 1$. Our assumption is that the direct calculation of $R(t)$ becomes more difficult the closer we get to 0, and thus, the target points z_i are smaller than the test points y_i, which, in turn, are smaller than the input points x_i:

$$z_1 < z_2 < \cdots < z_s < y_1 < y_2 < \cdots < y_l < x_1 < x_2 < \cdots < x_k$$

To check the validity of the RI approach, we calculated a $(5, 5)$ RI based on the simulated reliabilities at the input points $t = 13, ..., 23$ and a logarithmic transformation. We then used the obtained function to predict the reliabilities $\hat{R}_{5,5}(t)$ for $t = 0, ..., 50$.

Both the exact and the predicted reliabilities are depicted in Figure 2 and it is clear that the difference between the two is very small. Figure 3 shows the exact values of these differences in log-scale. Note that the differences are especially small for the input points. They are not equal to zero because the curve was based on simulation results while it was compared to the exact analytical results.

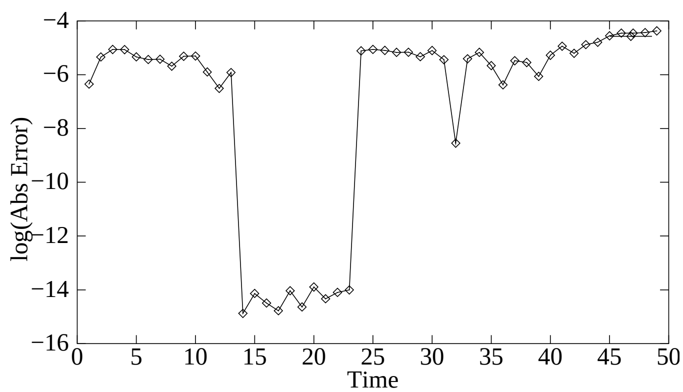

Figure 3. The difference between the exact and the predicted reliabilities (log-scale)

In more complex systems, an analytic solution will not be available to us and we will have to resort to simulation, for which a good random number generator is a must. It is well known that there isn't one random number generator which is suitable for all tasks. We tried several random numbers generators and finally selected the Mersenne Twister generator [14]. Figure 4 shows the average difference over the points $t = 0, ..., 50$ when comparing the exact reliability $R(t)$ to the simulated reliability $R^s(t)$ as a function of the number of simulations performed using this generator.

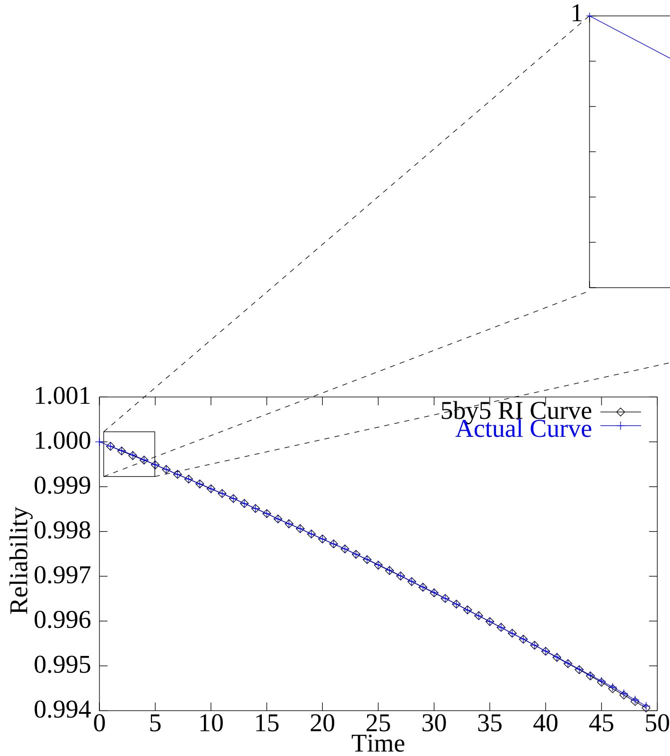

Figure 2. Comparing actual and RI reliabilities.

To increase the precision of our numerical calculations, we used the multi-precision software developed by Bailey et al. [15] and available at [16], and noticed a significance increase in accuracy compared to earlier computations.

In the next set of numerical experiments, we chose as target points $z_1, ..., z_s = 1, 2, 3, 4, 5$.
Our first experiment was meant to assess the sensitivity of the best RI for the target points to the selection of the input points. For this experiment, we didn't use any test points but varied the placement and number of the input points. For each starting point between 6 and 20 we calculated all possible RIs between $(1, 1)$ and $(10, 10)$, based on the analytic reliability function. We then calculated the estimates for the target points $1, ..., 5$ and the average error $\bar{E}_{m,n}(Z)$. We then listed those RIs for which $\bar{E}_{m,n}(Z) < 10^{-12}$. The results of this experiment are reported in Table 1, and they confirm that selecting input points which are closer to the target points yields higher precision and a larger number of possible RIs which provide this precision.

We repeated this experiment with the simulated reliabilities, and due to simulation noise we listed those RIs for which $\bar{E}_{m,n}(Z) < 10^{-7}$. The results appear in Table 2, and they show that for simulated data the effect of the input points positioning is less prominent.

In the second experiment, we fixed both the number and the placement of the test points and the target points, but varied the number and the placement of the input points, and consequently, the degree of the RI approximation. The target points were again $1, ..., 5$ and the test points were $6, ..., 12$. The input points were $13, 14, ..., 50$ with the starting point varying from 13 to 20, and the degree of the RI varying from $(1, 1)$ to $(10, 10)$. The purpose of this experiment was to see whether there exists a correspondence between $\bar{E}_{m,n}(Y)$ and $\bar{E}_{m,n}(Z)$ for varying combinations of starting point and (m, n). Such a correspondence would indicate that selecting the best RI for the test points is very likely to result in a good fit for the target points as well. Again, this experiment was performed for both the accurate and the simulated input points.

We first calculated the correlation coefficient between $\bar{E}_{m,n}(Y)$ and $\bar{E}_{m,n}(Z)$ for each starting point between 13 and 20, for both the exact and the simulated input points. In all cases the correlation was about 0.99. This very high correlation results from the fact that (m, n) RIs which produced very large errors in the test points did the same in the target points. We, therefore, restricted the calculation to RIs which had $\bar{E}_{m,n}(Y) < 10^{-7}$, and the resulting correlations for the different placements were around 0.95 for the exact input points and slightly lower, around 0.85 for the simulated inputs.

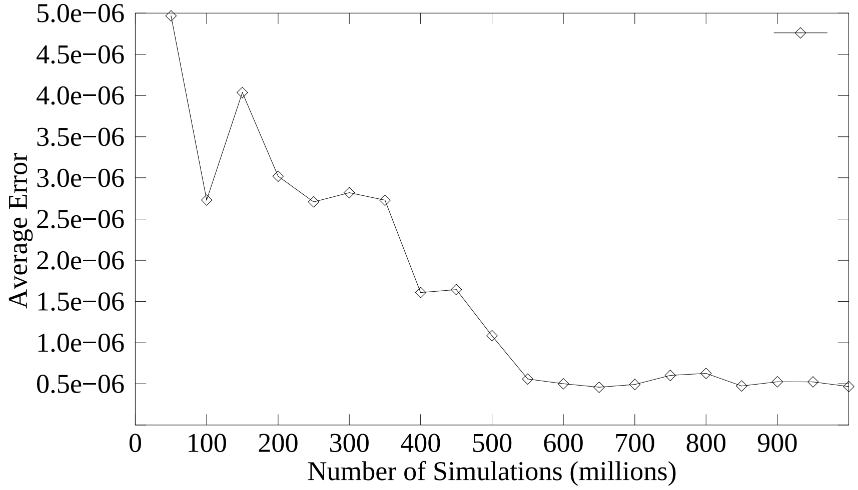

Figure 4. Comparing the simulated reliability to the analytic results.

We then compared, for each starting point, the set of best RIs for both sets of input data. The results are reported in Tables 3 and 4. Our conclusion is that an RI which shows a good fit to the test points is very likely to have a good fit to the target points as well.

In the third experiment, we wanted to find the effect of the simulation time on the accuracy of the reliability estimates. This is important in case of a restricted simulation budget, enabling us to determine how many input points should be used and for how long should the simulation last. We simulated the system reliability at the input points $t = 13, 14, ...$ and used the results to get two RIs: $(7, 5)$ and $(8, 6)$. For both RIs we estimated $\hat{R}(t)$ at the target points $1, ..., 5$ and calculated the average error compared to the actual reliabilities at these points. This was done for a sequence of simulation times, and the results are depicted in Figure 5. Clearly, the longer the simulation time, the less noisy are the results and the better is the interpolation. However, it seems that we are not getting much added accuracy by simulating longer than 250 million cycles.

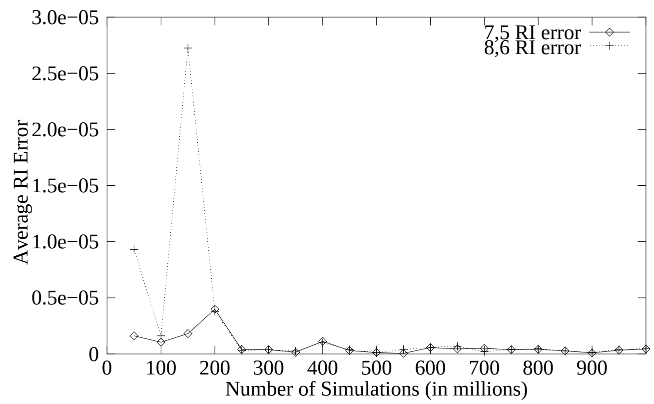

Figure 5. Target points error as a function of the simulation time.

5. Discussion

In this paper we have reported a case study in the use of Rational Interpolations for calculating reliabilities which are very close to 1. We have demonstrated the usefulness of a technique for selecting an accurate RI. This approach is designed to provide results in cases when a brute case approach is not feasible.

We are currently exploring the use of this technique for the analysis of more complicated systems and for the generation of preemptive maintenance strategies.

6. Acknowledgment

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Starting Point	M,Ns that satisfy the error constraint
6	(10,9),(7,6),(5,4),(4,5),(9,8),(8,7),(6,7),(9,10),(6,5),(8,9),(7,8),(6,3),(5,6),(9,9),(10,8),(8,10),(8,5),(7,4),(10,7),(9,6),(9,7),(7,9),(8,8),(7,5),(6,6),(5,7),(7,7),(8,6),(6,8),(4,3),(3,4),(7,10),(5,8),(6,9),(10,10),(4,4),(5,9),(5,3),(3,5),(2,2),(3,3),(4,6),(2,6),(3,2),(2,3),(4,2),(3,1),(6,10),(6,4),(2,7)
7	(8,9),(7,8),(9,8),(9,10),(10,9),(10,10),(9,9),(9,10),(8,7)(10,8),(5,6),(6,5),(6,7),(10,7),(7,6),(7,9),(9,7),(8,8),(8,5),(9,6),(5,7),(7,5),(6,6),(6,8),(5,2),(6,3),(8,6),(7,7),(3,4),(4,3),(2,2),(7,10),(3,1),(3,5),(4,6),(6,9)
8	(10,9),(9,8),(8,10),(10,8),(9,9),(10,10),(6,5),(9,10),(8,9),(8,7),(5,6),(7,8),(7,6),(6,7),(4,3),(3,4),(5,4),(8,8),(9,7),(7,9),(4,5),(3,5),(7,7),(8,6),(6,8),(7,10),(5,3),(5,5),(6,4)
9	(9,6),(7,8),(8,5),(10,9),(8,7),(9,10),(6,7),(7,6),(6,3),(7,9),(9,7),(8,10),(8,8),(10,8),(9,9),(10,10),(4,5),(7,10),(5,4),(2,2),(10,7),(8,9),(7,4)
10	(7,9),(9,7),(8,8),(9,8),(8,7),(10,9),(8,9),(9,10),(7,8),(8,10),(9,9),(10,8),(10,10),(5,10),(7,6)
11	(9,10),(8,5),(8,9),(9,8),(6,7),(10,9),(7,6),(2,7),(7,4),(5,6)
12	(6,8),(8,6),(7,7),(8,7),(7,6),(9,8),(7,8),(8,9),(6,7),(8,8),(7,9)
13	(5,5),(6,4),(4,6),(4,3),(7,8),(3,4),(7,6),(5,2)
14	(6,6),(7,5),(5,7),(7,6),(6,5),(9,10),(8,7),(6,7),(5,6),(7,8)
15	(8,10),(10,8),(9,9),(7,8),(8,9),(9,10),(6,7),(10,10),(7,6),(6,5)
16	(8,9),(9,10)
17	(9,8)
18	(10,10),(8,9),(7,8)
19	(7,5),(5,7),(6,6),(4,3),(4,5)
20	(9,9),(8,10),(10,8),(7,8),(6,7)

Table 1: Impact of input points placement: exact values.

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Starting Point	M,Ns that satisfy the error constraint
6	(4,0),(1,3),(1,1),(2,0),(3,7),(3,4)
7	(3,1),(10,9),(1,1),(2,2),(8,9),(5,3),(9,10),(8,6),(9,7),(2,0), (10,7),(9,8),(10,8),(7,8),(9,6),(4,5),(8,7),(6,3),(7,5),(5,4), (10,5),(6,4),(5,6)
8	(5,4),(4,5),(6,5),(9,6),(10,7),(5,6),(10,8),(8,6),(9,7),(7,5), (6,4),(4,2),(5,3),(3,1),(9,8),(10,9),(6,3),(8,7),(8,9),(9,10), (7,6),(7,8),(6,7),(9,9),(10,10),(8,10),(10,6),(4,3),(5,2),(3,4), (2,0),(8,8),(7,7),(2,2),(4,4),(3,3),(5,5)
9	(10,9),(1,1),(9,8),(10,7),(9,10),(4,2),(8,9),(7,5),(8,6),(2,0), (10,8),(9,7),(7,8),(6,4),(5,3),(9,6),(8,7)
10	(7,3),(10,7),(10,8),(8,6),(7,5),(4,6),(4,2),(5,3)
11	(9,8),(9,10),(10,7),(8,7),(9,6),(8,9),(3,1),(7,8),(6,4),(10,8), (7,5),(10,9),(9,7),(8,6),(5,3),(4,2)
12	(1,1),(4,4),(3,3)
13	(7,6),(8,5),(8,7),(9,10),(8,9),(7,8),(9,6),(2,0),(6,7),(5,3), (9,7),(6,4),(10,9),(10,8),(8,6),(7,5),(10,7),(9,8),(4,2),(3,1)
14	(7,5),(8,6)
15	(5,6),(10,8),(6,7),(4,2),(8,6),(5,3),(8,9),(9,7),(7,4),(9,10), (6,5),(7,5),(6,4),(7,6),(7,8),(10,10),(8,5),(3,1),(9,8),(2,0)
16	(1,1),(3,1)
17	(6,3),(5,4),(5,6),(7,8),(10,8),(8,9),(7,7),(7,4),(4,5),(6,5), (9,7),(3,1),(4,2),(6,7),(7,5),(8,6),(10,10),(6,4),(5,3),(9,6), (8,7),(9,10),(10,7)
18	(1,1),(5,3)
19	(4,3),(10,6),(5,2),(6,7),(6,3),(10,8),(5,4),(7,8),(9,7),(4,5), (2,0),(8,6),(5,6),(3,4),(3,1),(6,4),(7,5),(9,9),(5,3),(4,2), (8,5),(7,6),(9,10),(8,9),(9,6)
20	(8,5),(4,2)

Table 2: Impact of input points placement: simulation values.

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Start	M,Ns(test points)	M,Ns (target Points)	% match	Error<=
13	(2,4),(7,6),(4,3),(3,5),(3,4),(5,5),(6,4),(4,6),(4,2),(5,2),(10,8),(8,10),(3,3),(5,3),(6,7),(3,2),(2,3)	(5,5),(6,4),(4,6),(4,3),(7,8)	80.00	3.00E-013
14	(6,6),(7,5),(5,7)	(7,5),(6,6),(5,7)	100.00	1.00E-013
15	(8,10),(9,9),(6,7),(7,8),(9,10),(8,9)	(8,10),(9,9),(7,8),(9,10),(8,9)	100.00	1.00E-013
16	(8,9),(9,10)	(8,9),(9,10)	100.00	4.00E-013
17	(9,8),(3,5),(5,3),(4,4)	(9,8)	100.00	6.00E-013
18	(8,9),(7,8)	(8,9),(7,8)	100.00	5.00E-013
19	(8,5),(7,4),(4,3),(6,9),(6,3),(7,5),(5,7),(6,6),(4,5),(3,4)	(7,5),(5,7),(6,6),(4,3),(4,5),(9,6)	66.66	7.00E-013
20	(9,9),(8,10)	(9,9),(8,10)	100.00	2.00E-013

Table 3: Error correlation between test and target points: exact values.

Start	M,Ns(test points)	M,Ns (target Points)	% match	Error<=
13	(7,5),(8,6),(10,8),(9,7),(6,4),(5,3),(6,7),(4,2),(7,8),(2,0),(3,1),(8,5),(10,10),(9,10),(1,1),(8,7)	(7,6),(8,5),(8,7),(9,10),(8,9),(7,8),(9,6),(2,0),(6,7),(5,3),(9,7),(6,4),(10,9),(10,8),(10,6),(7,5),(10,7),(9,8),(4,2)	70.50	$4.00E-007$
14	(7,5),(8,6)	(7,5),(8,6)	100.00	$5.00E-007$
15	(3,1),(6,4),(7,5),(2,0),(10,10),(9,7),(9,9),(5,3),(5,3),(8,6),(4,2),(7,7),(10,8),(5,6)	(5,6),(10,8),(6,7),(4,2),(8,6),(5,3) (8,9),(9,7),(7,4),(9,10),(6,5),(7,5),(6,4),(7,6),(7,8),(10,10), (8,5),(3,1),(9,8),(2,0)	76.92	$4.82E-007$
16	(2,2),(1,1),(2,4),(6,4)	(1,1),(3,1),(3,3),(6,4),(2,2),(2,0),(7,5),(9,10)	75.00	$9.00E-007$
17	(6,4),(8,6),(5,3),(7,5),(4,2),(9,7),(10,10),(3,1),(4,5),(9,9)	(6,3),(5,4),(5,6),(7,8),(10,8),(8,9),(7,4),(4,5),(6,5),(9,7),(3,1),(4,2),(6,7),(7,5),(8,6),(10,10),(6,4),(5,3)	90.00	$3.00E-007$
18	(5,3),(6,4)	(1,1),(5,3),(6,4)	100.00	$7.00E-007$
19	(6,4),(8,6),(3,1),(3,4),(7,5),(5,3), (4,2),(2,0),(9,9),(9,7),(4,5),(8,10),(10,6),(5,2),(4,3),(8,8)	(4,3),(10,6),(5,2),(6,7),(6,3),(10,8),(5,4),(7,8),(9,7),(4,5),(2,0),(8,6),(5,6),(3,4),(3,1),(6,4),(7,5),(9,9),(5,3),(4,2),(8,5),(7,6)	75.00	$4.00E-007$
20	(8,5),(4,2),(5,3)	(8,5),(4,2),(5,3)	100.00	$5.23E-007$

Table 4: Error correlation between test and target points: simulation values.