# A New Approach to the Evaluation of the Reliability of Digital Systems

ISRAEL KOREN AND EITAN SADEH

*Abstract*—**Signal reliability, as a measure of digital systems' reliability, has not been used until recently due to lack of efficient evaluation methods. A new approach to the evaluation of signal reliability is presented in this work. A reliability transfer function of digital systems is defined and a method for its evaluation is presented. This approach provides a new insight into the problem of digital system reliability. Furthermore, it simplifies signal reliability calculations and can easily be mechanized.**

*Index Terms*—**Functional reliability, multiple faults, reliability transfer matrix, star product, signal reliability.**

## I. Introduction

Two reliability measures can be employed when the reliability of a digital system is evaluated, namely, functional reliability and signal reliability [1]–[6]. The first one is undoubtedly simpler to apply and requires a smaller amount of computation however, it is known to be exceedingly pessimistic [1]–[6]. The more accurate signal reliability measure has not been used until recently, due to lack of efficient evaluation methods. Algorithms for the evaluation of signal reliabilities have been introduced lately by Ogus [2] and Koren [4], [6]. However, both methods require complex symbolic manipulations resulting from the existence of statistical dependence among the various signals in a digital system [2]–[4], [6].

In this work we present a new approach to the evaluation of signal reliabilities in which statistical dependence between signals is handled in a natural way and symbol manipulations are avoided. We introduce the concept of reliability transfer function, enabling us to incorporate any fault model into our analysis and to consider large subsystems (e.g., IC modules) rather than single gates as basic elements. Consequently, for a large class of digital systems this new approach reduces considerably the amount of computation involved in evaluating signal reliabilities.

## II. Preliminaries

To evaluate the reliability of a digital system we need to have some knowledge on the nature of the possible faults and their probabilities of occurrence. We assume that the possible faults are multiple lead failures (not necessarily permanent stuck-at faults) and we denote by $s$ the probability of a single lead failure. Since the faults on different lines in the system are not necessarily equiprobable we denote by $s_X$ the probability of a fault on line $X$. The probability $s$ is in general time-dependent and the most commonly used fault probability function is $s(t) = 1 - e^{-\lambda t}$ where $\lambda$ is the failure rate. Consequently, the signal reliability of the system, denoted by $SR(t)$, is time-dependent and is defined as follows:

$$SR(t) = \Pr\{\text{the output signal is correct at time } t\}.$$

In some applications of reliability analysis (e.g., prediction of mission time) the accumulative signal reliability in the time interval $[0, t]$ rather than at instant $t$, is needed. This accumulative signal reliability, denoted by $R_s(t)$, is defined as follows:

$$R_s(t) = \Pr\{\text{the output signals are correct in the time interval } [0, t]\}.$$

These two signal reliabilities have been analyzed and compared to the corresponding functional reliabilities [6]. Here we restrict ourselves to evaluation of the non-accumulative signal reliability and, for convenience, we call it signal reliability. Several applications of the signal reliability were mentioned in [2]–[4], [6], [7]. One of the important applications of signal reliability is comparison between different realizations of a logical system. Employing the functional reliability measure results in a less accurate reliability comparison of different designs. When the functional reliability is evaluated, the reliability of the basic element is raised to the number of these elements in the system, e.g., [5], [8]. Although reliability is a function of the complexity of the system, the complexity may not be treated as a simple function of the number of basic elements [8]. Contrary to the functional reliability measure, the signal reliability depends upon the exact structure of the system, the nature of the possible failures and their probabilities, thus yielding a more accurate reliability comparison of different designs.

In the next section we present a procedure for the evaluation of the signal reliability of combinational systems. For convenience, we omit $t$ as an argument of the reliability and failure probability functions and these functions are understood to be time-dependent.

## III. The Reliability Transfer Matrix

The presence of faults in a system may cause incorrect logic signals on some lines. Consequently, the signal on each line $X$ may assume one of four values, namely, correct 0, correct 1, incorrect 0 and incorrect 1. These values will be designated by 0, 1, 2 and 3, respectively. Thus, the signal $X$ is a random four-valued variable and the probabilities of its four possible values are

$$\Pr\{X = 0\} = \Pr\{X \text{ is correctly a } 0\} \triangleq R_0(X)$$

$$\Pr\{X = 1\} = \Pr\{X \text{ is correctly a } 1\} \triangleq R_1(X)$$

$$\Pr\{X = 2\} = \Pr\{X \text{ is incorrectly a } 0\} \triangleq R_2(X)$$

$$\Pr\{X = 3\} = \Pr|X \text{ is incorrectly a } 1\} \triangleq R_3(X).$$

Clearly, $R_0(X) + R_1(X) + R_2(X) + R_3(X) = 1$.

The signal reliability of line $X$, denoted $SR(X)$, is the probability that the signal on line $X$ is correct, hence,

$$\mathrm{SR}(X) = R_0(X) + R_1(X).$$

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The signal reliability of a system, whose output is $Y$, is $SR(Y)$. This reliability is calculated from the input lines' reliabilities $R_0$, $R_1$, $R_2$, and $R_3$ using the reliability model devised in [4]. In this model, the occurrence of faults is introduced through special elements called fault occurrence networks (FON's). Such an element is inserted into each line of the system. Faults may occur in these elements only, and the rest of the system is considered fault-free. Thus, an FON element can be viewed as a set of conditional probabilities $\Pr\{Y=j \mid X=i\}$; $i, j \in \{0, 1, 2, 3\}$ which is the probability that, given the input $X$ to the FON is $i$, the output $Y$ is $j$.

For a given system $M$ we calculate a reliability transfer function in a form of a matrix relating the output signal reliability to the input signal reliabilities. Let $X_1, X_2, \cdots X_n$ and $Y$ be the $n$ independent input variables and the output line of system $M$, respectively. We define the reliability vector $R(X)$ of line $X$ as $R(X) = (R_0(X), R_1(X), R_2(X), R_3(X))$. The reliability transfer function of the system $M$ thus relates the output reliability vector $R(Y)$ to the input reliability vectors $R(X_1), \cdots, R(X_n)$. This function is derived in the following way. Let $\bar{X}$ denote the input vector $X_1, X_2, \cdots, X_n$ and $i = (i_1, i_2, \cdots, i_n)$ denote a specific four-valued vector assumed by $\bar{X}$. Each element of $R(Y)$ can be expressed as follows:

$$R_j(Y) = \Pr\{Y=j\} = \sum_{\substack{\text{all four-valued}\\ \text{vectors } i}} \Pr\{Y=j \mid \bar{X}=i\} \cdot \Pr\{\bar{X}=i\}; \qquad j = 0, 1, 2, 3.$$

The sum is over all $4^n$ four-valued vectors of length $n$; $i = (i_1, i_2, \cdots, i_n)$; $i_k = 0, 1, 2, 3$. To simplify notation, $i$ will be used to denote a four-valued vector and its decimal value interchangeably. Hence,

$$R_j(Y) = \sum_{i=1}^{4^n-1} \Pr\{Y=j \mid \bar{X}=i\} \cdot \Pr\{X_1=i_1, X_2=i_2, \cdots, X_n=i_n\}. \tag{3.1}$$

Since the input variables are independent,

$$\Pr\{X_1=i_1, X_2=i_2, \cdots, X_n=i_n\} = \prod_{k=1}^{n} \Pr\{X_k=i_k\} = \prod_{k=1}^{n} R_{i_k}(X_k). \tag{3.2}$$

To simplify notation, let this product term be denoted by $V_i(\bar{X})$, and let $t_{ij}$ denote the conditional probability $\Pr\{Y=j \mid \bar{X}=i\}$. Using this notation, we obtain from (3.1) and (3.2)

$$R_j(Y) = \sum_{i=0}^{4^n-1} t_{ij} V_i(\bar{X}). \tag{3.3}$$

The probabilities $t_{ij}$; $i = 0, 1, \cdots, 4^n-1$; $j = 0, 1, 2, 3$ form a stochastic matrix $T = \{t_{ij}\}$ of order $4^n \times 4$. The product terms $V_i(\bar{X})$; $i = 0, 1, \cdots, 4^n-1$ form a vector $V(\bar{X})$ of length $4^n$. Thus, (3.3) takes on the following matrix form

$$R(Y) = V(\bar{X}) \cdot T. \tag{3.4}$$

$T$ is called the reliability transfer matrix, abbreviated RTM. The size $4^{n+1}$ of the RTM increases rapidly with $n$. In Section V we show that only a reduced matrix of size $\sqrt{4^{n+1}}$ is actually needed.

An RTM can also be defined for multiple-output systems. Let $\bar{Y} = Y_1, Y_2, \cdots, Y_m$ be the output vector of a system. Let $W_j(\bar{Y})$ denote the probability that $\bar{Y}$ equals $j$, i.e.,

$$W_j(\bar{Y}) = \Pr\{\bar{Y}=j\}; \qquad j = 0, 1, \cdots, 4^m-1.$$

These elements form a vector $W(\bar{Y})$, and are calculated as follows:

$$W_j(\bar{Y}) = \sum_{i=0}^{4^n-1} \Pr\{\bar{Y}=j \mid \bar{X}=i\} \cdot \Pr\{\bar{X}=i\}.$$

The conditional probability $\Pr\{\bar{Y}=j \mid \bar{X}=i\}$ is the element $t_{ij}$ of the RTM of the multiple-output system, i.e., $T = \{t_{ij}\}$; $i = 0, 1, \cdots, 4^n-1$; $j = 0, 1, \cdots, 4^m-1$. Therefore,

$$W_j(\bar{Y}) = \sum_{i=0}^{4^n-1} \Pr\{\bar{X}=i\} \cdot t_{ij} = \sum_{i=0}^{4^n-1} V_i(\bar{X}) \cdot t_{ij}.$$

Hence,

$$W(\bar{Y}) = V(\bar{X}) \cdot T. \tag{3.5}$$

To reduce the complexity of the evaluation of the RTM, the given system is decomposed into subsystems and an appropriate RTM is calculated for each subsystem. The RTM of the overall system is then calculated using the RTM's of the subsystems. The smallest subsystems considered are the basic elements of the model. In the following we derive the RTM's of these basic elements.

*FON*: Let $X$, $Y$ be the input and output lines of an FON, respectively. The elements of the RTM $T_{FON}$ depend upon the types of faults assumed to occur at line $X$. If the possible faults are stuck-at-zero (s-a-0) and stuck-at-one (s-a-1) with probabilities $q_{0_x}$ and $q_{1_x}$, respectively, satisfying $s_x = q_{0_x} + q_{1_x}$, then the elements of the RTM are

$$\begin{aligned} t_{00} &= \Pr\{Y=0 \mid X=0\} \\ &= \Pr\{Y \text{ is correctly a } 0/X \text{ is correctly a } 0\} \\ &= \Pr\{\text{No s-a-1 fault occurred}\} = 1 - q_{1_x} \\ t_{01} &= \Pr\{Y=1 \mid X=0\} \\ &= \Pr\{Y \text{ is correctly a } 1/X \text{ is correctly a } 0\} = 0. \end{aligned}$$

In a similar manner, $t_{02} = 0$.

$$\begin{aligned} t_{03} &= \Pr\{Y=3 \mid X=0\} \\ &= \Pr\{Y \text{ is incorrectly a } 1/X \text{ is correctly a } 0\} \\ &= \Pr\{\text{A s-a-1 fault occurred}\} = q_{1_x}. \end{aligned}$$

Similarly, all other elements of $T_{FON}$ are calculated, yielding

$$T_{FON} = \begin{bmatrix} 1-q_{1_x} & 0 & 0 & q_{1_x} \\ 0 & 1-q_{0_x} & q_{0_x} & 0 \\ 0 & q_{1_x} & 1-q_{1_x} & 0 \\ q_{0_x} & 0 & 0 & 1-q_{0_x} \end{bmatrix}. \tag{3.6}$$

If the possible fault is an "inverted signal" fault (i.e., $Y = X'$) with probability $s_x$, the resulting RTM is

$$T_{FON} = \begin{bmatrix} 1-s_x & s_x & 0 & 0 \\ s_x & 1-s_x & 0 & 0 \\ 0 & 0 & 1-s_x & s_x \\ 0 & 0 & s_x & 1-s_x \end{bmatrix}.$$

In both cases the lead failures are not necessarily permanent. If the fault at line $X$ is permanent then $s_x(t) = 1 - \exp(-\lambda_x t)$. If it is intermittent then $s_x(t)$ is the probability that the intermittent fault is in the active state at time $t$. The exact expression for $s_x(t)$ depends upon the model selected for the intermittent fault, e.g., [9], [10].

For the various leads in the system different faults may be assumed. Some of the leads may be fault-free (i.e., $s_x = 0$) yielding $T_{FON} = I$.

NOT *Gate*: Let $X$, $Y$ be the input lines of a fault-free NOT gate.

Then

$$t_{00} = \Pr\{Y = 0 \mid X = 0\}$$
$$= \Pr\{Y \text{ is correctly a } 0/X \text{ is correctly a } 0\} = 0$$
$$t_{01} = \Pr\{Y = 1 \mid X = 0\}$$
$$= \Pr\{Y \text{ is correctly a } 1/X \text{ is correctly a } 0\} = 1$$
$$t_{02} = \Pr\{Y = 2 \mid X = 0\}$$
$$= \Pr\{Y \text{ is incorrectly a } 0/X \text{ is correctly a } 0\} = 0$$
$$t_{03} = \Pr\{Y = 3 \mid X = 0\}$$
$$= \Pr\{Y \text{ is incorrectly a } 1/X \text{ is correctly a } 0\} = 0.$$

Besides $t_{01}$, the other nonzero elements of $T_{\text{NOT}}$ are $t_{10} = t_{23} = t_{32} = 1$. Thus

$$T_{\text{NOT}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

*Basic Gates:* Let $\bar{X} = X_1, X_2, \cdots, X_n$ be the independent input lines of a gate whose output line is $Y$. Each of the commonly used gates (AND, OR, NAND, NOR) can be uniquely described by a binary vector $(\alpha_1, \alpha_2, \cdots, \alpha_n, \beta)$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ is the only input combination for which the output $Y$ equals $\beta$, e.g., a three-input NAND gate is described by the vector (1, 1, 1, 0). Using this describing vector the equations relating the output reliability vector to the input reliability vectors are [4] as follows:

$$R_\beta(Y) = \prod_{k=1}^{n} R_{\alpha_k}(X_k) \tag{3.7}$$

$$R_{\beta+2}(Y) = \prod_{k=1}^{n} [R_{\alpha_k+2}(X_k) + R_{\alpha_k}(X_k)] - R_\beta(Y) \tag{3.8}$$

$$R_{3-\beta}(Y) = \prod_{k=1}^{n} [R_{3-\alpha_k}(X_k) + R_{\alpha_k}(X_k)] - R_\beta(Y) \tag{3.9}$$

$$R_{1-\beta}(Y) = 1 - [R_\beta(Y) + R_{\beta+2}(Y) + R_{3-\beta}(Y)]. \tag{3.10}$$

By comparing these equations to (3.3) the elements of the matrix $T$ are derived. From (3.7) and (3.3) we have

$$R_\beta(Y) = \prod_{j=1}^{n} R_{\alpha_j}(X_j) = V_{\alpha_1, \alpha_2, \cdots, \alpha_n}(X) = \sum_{i=0}^{4^n-1} V_i(\bar{X}) \cdot t_{i\beta}.$$

Consequently,

$$t_{i\beta} = \begin{cases} 1 & \text{if } i = (\alpha_1, \alpha_2, \cdots, \alpha_n) \\ 0 & \text{otherwise} \end{cases}. \tag{3.11}$$

From (3.8) and (3.3) we have

$$t_{i,\beta+2} = \begin{cases} 1 & \text{if } i = (i_1, i_2, \cdots, i_n) \text{ and } i_k \in (\alpha_k, \alpha_k + 2); \\ & k = 1, 2, \cdots, n \text{ and } \exists m(i_m = \alpha_m + 2) \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

In a similar way

$$t_{i,3-\beta} = \begin{cases} 1 & \text{if } i = (i_1, i_2, \cdots, i_n) \text{ and } i_k \in (\alpha_k, 3 - \alpha_k); \\ & i = 1, 2, \cdots, n \text{ and } \exists m(i_m = 3 - \alpha_m) \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Finally,

$$t_{i,1-\beta} = 1 - (t_{i,\beta} + t_{i,\beta+2} + t_{i,3-\beta}). \tag{3.14}$$

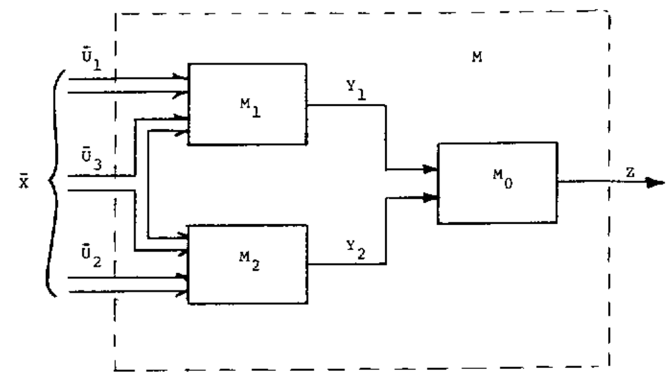


Fig. 1. A system constructed of three subsystems.

## IV. Calculations of Reliability Transfer Matrices

The RTM of a given system is calculated from the RTM's of its components which are either basic elements or subsystems whose RTM's are known. The operation used in the construction of the system's RTM is called the star product and is defined below.

Let $A^{(1)}$, $A^{(2)}$ be the RTM's of the two subsystems $M_1$, $M_2$ in Fig. 1, respectively. Let $\bar{X} = (\bar{U}_1, \bar{U}_2, \bar{U}_3)$ be the input vector to the system $M$, where $\bar{U}_1$, $\bar{U}_2$ are the subsets of input lines feeding solely $M_1$, $M_2$, respectively, and $\bar{U}_3$ is the subset of input lines in common to $M_1$ and $M_2$. Let $n_i$ be the cardinality of the subset $\bar{U}_i$, i.e., $n_i = |U_i|$, then

$$n = |\bar{X}| = n_1 + n_2 + n_3.$$

Consequently, the dimensions of the matrices $A^{(1)}$, $A^{(2)}$ are $(4^{n_1+n_3} \times 4)$, $(4^{n_2+n_3} \times 4)$, respectively.

*Definition 4.1:* The star product of the matrices $A^{(1)} = \{a_{ij}^{(1)}\}$ and $A^{(2)} = \{a_{ij}^{(2)}\}$ is a matrix $C = A^{(1)} \circledast A^{(2)}$ of dimensions $4^n \times 4^2$ whose elements are

$$c_{ij} = a_{i_1,j_1}^{(1)} a_{i_2,j_2}^{(2)} \tag{4.1}$$

where

$$i_1 = \left[\frac{i}{4^{n_2+n_3}}\right] + i \bmod (4^{n_3}); \qquad j_1 = [j/4]$$

($[X]$ is the integer part of $X$.)

$$i_2 = i \bmod (4^{n_2+n_3}) \qquad j_2 = j \bmod (4).$$

In the special case where $n_3 = 0$, i.e., the subsets of input lines to $M_1$ and $M_2$ are disjoint, we have

$$n = n_1 + n_2$$

$$i_1 = [i/4^{n_2}];\ i_2 = i \bmod (4^{n_2});\ j_1 \text{ and } j_2 \text{ remain unchanged.}$$

The star product, in this case, reduces to the Kronecker product [11], yielding

$$A^{(1)} \circledast A^{(2)} = \begin{bmatrix} a_{00}^{(1)} A^{(2)} & a_{01}^{(1)} A^{(2)} & a_{02}^{(1)} A^{(2)} & a_{03}^{(1)} A^{(2)} \\ a_{10}^{(1)} A^{(2)} & & \cdots & \\ \vdots & & & \\ a_{4^{n_1}-1,0}^{(1)} A^{(2)} & & \cdots & a_{4^{n_1}-1,3}^{(1)} A^{(2)} \end{bmatrix}.$$

The definition of the star product can be generalized to multiple-output RTM's in the following manner. Let $\bar{W} = Y_{11}, Y_{12}, \cdots, Y_{1m}$; $\bar{V} = Y_{21}, Y_{22}, \cdots, Y_{2r}$ be the output vectors of $M_1$ and $M_2$, respectively. The input vectors to these subsystems are $(\bar{U}_1, \bar{U}_3)$ and $(\bar{U}_2, \bar{U}_3)$. The dimensions of the matrices $A^{(1)}$ and $A^{(2)}$ are now $(4^{n_1+n_3} \times 4^m)$ and $(4^{n_2+n_3} \times 4^r)$, respectively.

The star product $C = A^{(1)} \circledast A^{(2)}$ is a matrix of dimensions $4^n \times 4^{m+r}$ whose elements are

$$c_{ij} = a^{(1)}_{i_1, j_1} a^{(2)}_{i_2, j_2}$$

where $i_1$ and $i_2$ are the same as in Eq. (4.1) and

$$j_1 = [j/4^r];\ j_2 = j \bmod (4^r). \tag{4.2}$$

The star product is used next to calculate $T$, the RTM of the system $M$ in Fig. 1. Let $B^{(2)} = \{b^{(2)}_{ij}\}$ denote the RTM of the two-output system consisting of $M_1$ and $M_2$ and $A^{(0)}$ denote the RTM of $M_0$. The matrices $B^{(2)}$ and $T$ are derived in the following lemma.

*Lemma 4.1:*

$$B^{(2)} = A^{(1)} \circledast A^{(2)}$$

$$T = B^{(2)} \cdot A^{(0)} = (A^{(1)} \circledast A^{(2)})A^{(0)}.$$

*Proof:* $b^{(2)}_{ij} = \Pr\{\bar{Y} = j \mid \bar{X} = i\} = \Pr\{Y_1 = j_1,\ Y_2 = j_2 \mid \bar{X} = i\}$ where $j_1 = [j/4]$ and $j_2 = j \bmod (4)$. This conditional probability can be written as follows:

$$b^{(2)}_{ij} = \Pr\{Y_1 = j_1 \mid Y_2 = j_2,\ \bar{X} = i\} \cdot \Pr\{Y_2 = j_2 \mid \bar{X} = i\}.$$

Since the value of $Y_2$ is determined by the value of $\bar{X}$, the condition $Y_2 = j_2$ in the first term is redundant. Therefore,

$$b^{(2)}_{ij} = \Pr\{Y_1 = j_1 \mid \bar{X} = i\} \cdot \Pr\{Y_2 = j_2 \mid \bar{X} = i\}.$$

$Y_1$ depends on the values of the input lines $(\bar{U}_1, \bar{U}_3)$ to $M_1$ and $Y_2$ depends on the values of the input lines $(\bar{U}_2, \bar{U}_3)$ to $M_2$. Hence, $b^{(2)}_{ij} = \Pr\{Y_1 = j_1 \mid (\bar{U}_1,\ \bar{U}_3) = i_1\} \cdot \Pr\{Y_2 = j_2 \mid (\bar{U}_2,\ \bar{U}_3) = i_2\}$ where $i_1$ and $i_2$ are computed in the following way. The decimal value of the four-valued vector $\bar{X} = (\bar{U}_1, \bar{U}_2, \bar{U}_3)$ is $i$, thus the decimal value of $(\bar{U}_1, \bar{U}_3)$ is given by

$$i_1 = [i/4^{n_2+n_3}] + i \bmod (4^{n_3}).$$

Similarly, $i_2$ is the decimal value of $(\bar{U}_2,\ \bar{U}_3)$, i.e., $i_2 = i \bmod (4^{n_2+n_3})$. Consequently,

$$b^{(2)}_{ij} = a^{(1)}_{i_1, j_1} a^{(2)}_{i_2, j_2}.$$

Therefore, by Definition 4.1, $B^{(2)} = A^{(1)} \circledast A^{(2)}$. To prove the second part of the lemma, note that,

$$t_{ij} = \Pr\{Z = j \mid \bar{X} = i\}$$
$$= \sum_{k=0}^{4^2-1} \Pr\{Z = j \mid \bar{Y} = k,\ \bar{X} = i\} \cdot \Pr\{\bar{Y} = k \mid \bar{X} = i\}.$$

The value of $Z$ depends on the values of the $Y$ inputs only, therefore, the condition $\bar{X} = i$ in the first term is redundant:

$$t_{ij} = \sum_{k=0}^{15} \Pr\{Z = j \mid \bar{Y} = k\} \cdot \Pr\{\bar{Y} = k \mid \bar{X} = i\}$$
$$= \sum_{k=0}^{15} b^{(2)}_{ik} a^{(0)}_{kj}.$$

Hence, $T = B^{(2)} \cdot A^{(0)}$. Q.E.D.

To generalize the results of Lemma 4.1 for the case where $l$ subsystems $M_1, M_2, \cdots, M_l$ feed the subsystem $M_0$, consider the system depicted in Fig. 2. Let $A^{(0)}, A^{(1)}, \cdots, A^{(l)}$ denote the RTM's of the subsystems $M_0, M_1, \cdots, M_l$, respectively. Let $B^{(k)}$ denote the RTM of the $k$-output system consisting of $M_1, M_2, \cdots, M_k$ as shown in Fig. 3. This RTM is derived recursively using the following lemma.

*Lemma 4.2:* $B^{(k)} = B^{(k-1)} \circledast A^{(k)};\ k = 2, 3, \cdots, l.$

*Proof:* The lemma holds for $k = 2$ as proved in Lemma 4.1.

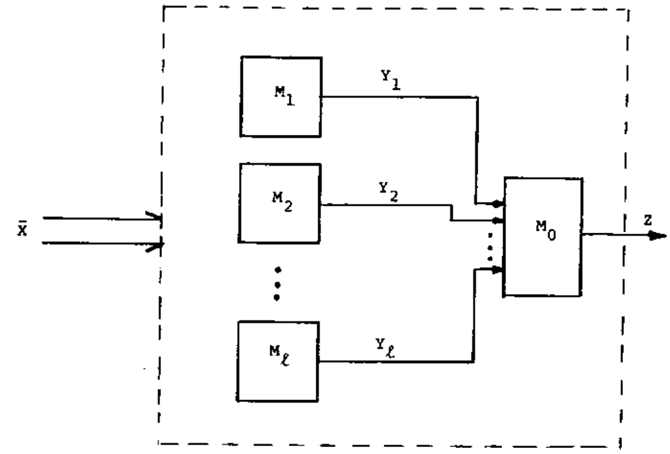


Fig. 2. A system constructed of $l + 1$ subsystems.

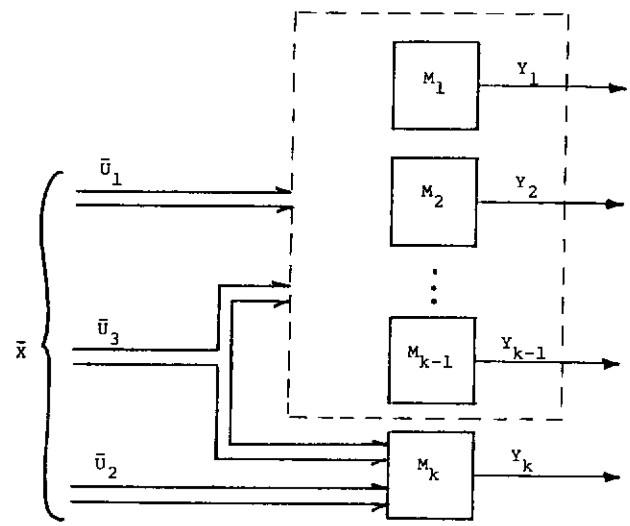


Fig. 3. A multiple output system constructed of $k$ subsystems.

Assuming that it is true for $k - 1$, we show that it is true for $k$:

$$b^{(k)}_{ij} = \Pr\{(Y_1, Y_2, \cdots, Y_k) = j \mid \bar{X} = i\}$$
$$= \Pr\{(Y_1, Y_2, \cdots, Y_{k-1}) = j_1,\ Y_k = j_2 \mid \bar{X} = i\}$$

where $j_1 = [j/4]$ and $j_2 = j \bmod (4)$.

Following the steps in the proof of Lemma 4.1, we obtain

$$b^{(k)}_{ij} = b^{(k-1)}_{i_1, j_1} \cdot a^{(k)}_{i_2, j_2}$$

where $i_1$ and $i_2$ are the same as in (4.1). Hence,

$$B^{(k)} = B^{(k-1)} \circledast A^{(k)}.$$ Q.E.D.

The RTM of the entire system in Fig. 2 is calculated in the following theorem.

*Theorem 4.1:* $T = B^{(l)} \cdot A^{(0)}$.

*Proof:*

$$t_{ij} = \Pr\{Z = j \mid \bar{X} = i\}$$
$$= \sum_{m=0}^{4^l-1} \Pr\{Z = j \mid \bar{Y} = m,\ \bar{X} = i\} \cdot \Pr\{\bar{Y} = m \mid \bar{X} = i\}$$
$$= \sum_{m=0}^{4^l-1} \Pr\{Z = j \mid \bar{Y} = m\} \cdot \Pr\{\bar{Y} = m \mid \bar{X} = i\}$$
$$= \sum_{m=0}^{4^l-1} a^{(0)}_{mj} \cdot b^{(l)}_{im} = \sum_{m=0}^{4^l-1} b^{(l)}_{im} a^{(0)}_{mj}.$$

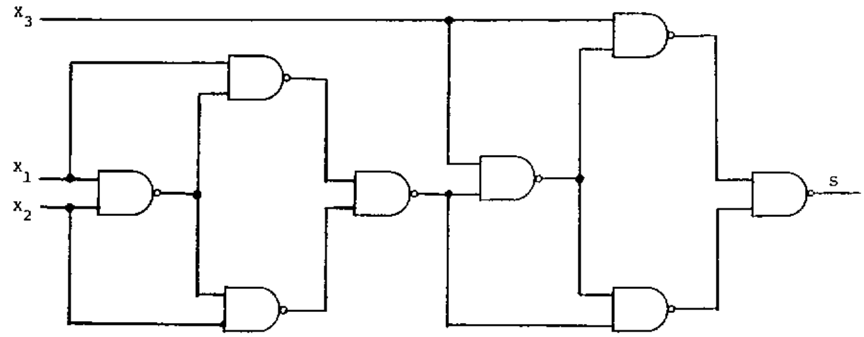


Fig. 4. An example system.

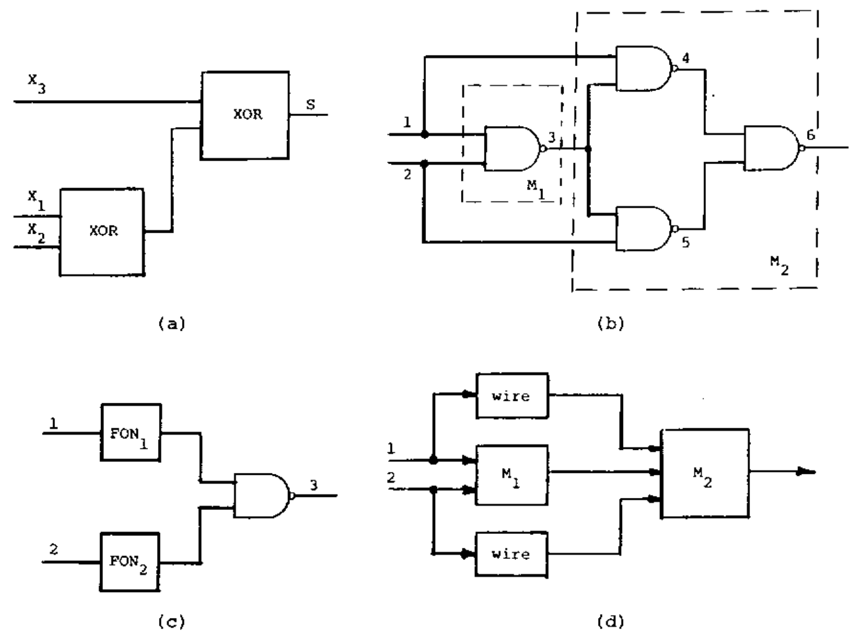


Fig. 5. Steps in the evaluation of the RTM.

Therefore,

$$T = B^{(l)}A^{(0)}. \quad \text{Q.E.D.}$$

*Corollary 4.1:* $B^{(k)} = (\cdots((A^{(1)} \circledast A^{(2)}) \circledast A^{(3)}) \circledast \cdots) \circledast A^{(k)}$.

*Corollary 4.2:* The star product is associative.

*Proof:* The proof is omitted for the sake of brevity.

*Corollary 4.3:* $T = (A^{(1)} \circledast A^{(2)} \circledast \cdots \circledast A^{(l)})A^{(0)}$.

The system in Fig. 2 for which Corollary 4.3 applies, has an internal fan-out-free (IFF) structure, i.e., only the input lines may fanout; the internal lines do not. In order to apply Corollary 4.3 to an arbitrary system, it must be partitioned into subsystems so that the resultant structure is an IFF one. This partitioning is applied recursively until a level is reached where the RTM's of the subsystems are known. In this process, each internal fan-out line eventually becomes an input line to some subsystem. This partitioning process, illustrated in the following example, can easily be mechanized.

*Example:* The RTM of the system in Fig. 4 is calculated as follows.

*Step 1:* The system is partitioned into two identical subsystems having the same RTM as shown in Fig. 5(a).

*Step 2:* To calculate the RTM of the XOR, it is further partitioned into subsystems $M_1$ and $M_2$ as shown in Fig. 5(b) so that fan-out line 3 becomes an input line to $M_2$.

*Step 3:* $M_1$ consists of two FON's and an NAND gate, as shown in Fig. 5(c). According to Corollary 4.3, the RTM of $M_1$ is given by

$$T_1 = (T_{FON_1} \circledast T_{FON_2})T_{NAND}.$$

In a similar manner, $T_2$ the RTM of $M_2$, can be derived.

*Step 4:* To calculate the RTM of the XOR subsystem, consider Fig. 5(d). Thus,

$$T_{XOR} = (I \circledast T_1 \circledast I)T_2.$$

*Step 5:* Finally, the RTM $T_S$ of the original system in Fig. 5(a) is given by

$$T_S = (I \circledast T_{XOR})T_{XOR}.$$

The method of calculating the RTM of a system using the RTM's of its subsystems is especially attractive in the following cases:

*a)* The system consists of standard LSI modules. Here, the RTM of a standard module used more than once throughout the system, has to be calculated just once.

*b)* The system consists of several identical subsystems, e.g., cellular arrays and NMR systems.

The second case is illustrated in the following example.

*Example:* Consider the TMR configuration of a full-adder shown in Fig. 6. The RTM of the TMR system $T_{TMR}$ is calculated

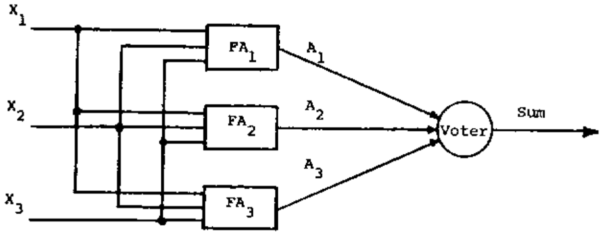

Fig. 6. A TMR configuration of a full adder.

using the RTM of a full-adder $T_{FA}$ and the RTM of the Majority Voter $T_V$ as follows

$$T_{TMR} = (T_{FA} \circledast T_{FA} \circledast T_{FA})T_V. \quad (4.3)$$

The extension of this equation to NMR systems is straightforward. For the TMR system (4.3) was used to calculate the improvement in the output signal reliability due to the TMR configuration. In this example we assume that the possible faults in the FA modules and the Voter are permanent stuck-at faults and that they are equally likely, i.e., $q_0 = q_1 = \frac{1}{2}s_{FA}$ for all lines in the FA and $q_0 = q_1 = \frac{1}{2}s_V$ for all lines in the Voter. We also assume that the input signals are either correct 0 or correct 1 with the same probability, i.e., $R_0(X_i) = R_1(X_i) = \frac{1}{2}$; $R_2(X_i) = R_3(X_i) = 0$; $i = 1, 2, 3$. The signal reliability of the sum output, SR(SUM), was computed using an APL program and compared to the reliability SR($A$). The improvement in the reliability depends upon the probability of a fault in the Voter. Hence, SR(SUM) was calculated first for $s_V = s_{FA}$ and then for $s_V = 0$. The results are plotted in Fig. 7 as a function of the lead fault probability in the FA, $s_{FA}$. The curve for $s_V = 0$ indicates the upper limit on the output signal reliability. To clarify the results, the percentage of improvement in reliability due to the TMR configuration is plotted in Fig. 8. The curve for $s_V = s_{FA}$ implies that the TMR configuration increases the reliability only for $s_{FA} \leq 0.093$ (or equivalently, for $t \leq (1/\lambda)\ 0.0976$ where $1/\lambda$ is the mean lifetime) and the maximal improvement in reliability is 4.4 percent. If $s_V = 0$ the TMR configuration always improves the reliability with maximal improvement of 12.1 percent.

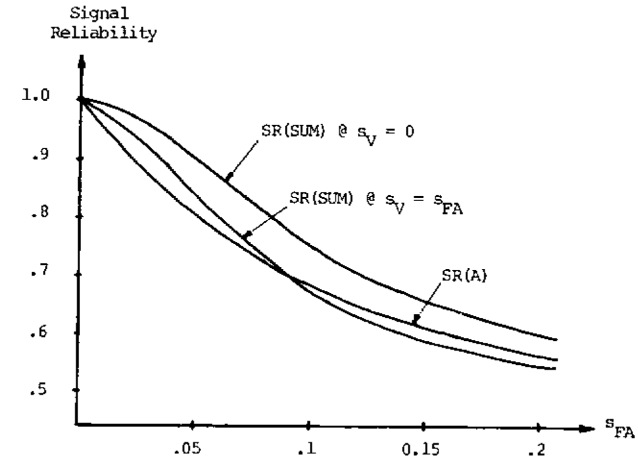

Fig. 7. The signal reliabilities of the TMR configuration in Fig. 6.

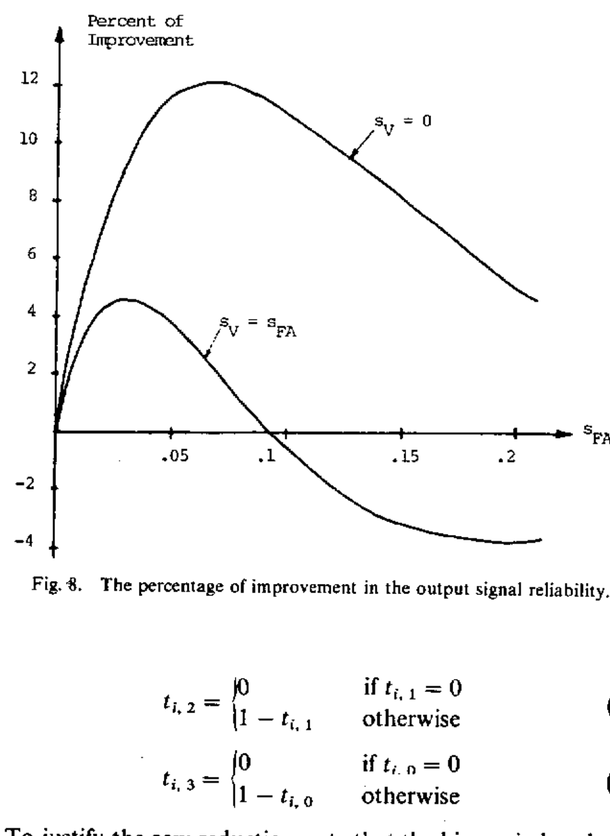

Fig. 8. The percentage of improvement in the output signal reliability.

## V. Reducing the Size of the RTM

The amount of computation and the size of computer memory needed when employing the previous method depend mainly upon the size of the RTM's of the standard modules. In the following we show that the size of the RTM can be considerably reduced and instead of $4^{n+1}$ entries only $\sqrt{4^{n+1}} = 2^{n+1}$ entries are needed. Specifically, the number of columns can be reduced from 4 to 2 and the number of rows from $4^n$ to $2^n$. Let $T^{(r)}$ denote the reduced matrix corresponding to an RTM $T$. In $T^{(r)}$ we include only the first two columns of $T$ and only the binary-indexed rows of $T$ where the $i$th row of $T$ is a binary-indexed row if the four-valued vector $i = (i_1, i_2, \cdots, i_n)$ is a binary vector, i.e., $i_k \in \{0, 1\}$ for $k = 1, 2, \cdots, n$.

We first justify the reduction of columns. In any row $i$ of $T$ exactly one out of the first two entries $t_{i,0}$ and $t_{i,1}$ equals 0 and exactly one out of the last two entries $t_{i,2}$ and $t_{i,3}$ equals 0. The reason is that the output of a module for a given input combination can be either 0 or 1 but not both. It is 0, then the actual output is either a correct 0 or an incorrect 1. Consequently, $t_{i,0}$ and $t_{i,3}$ are the only nonzero entries. Similarly, if the correct output is one then $t_{i,1}$ and $t_{i,2}$ are the only nonzero entries. Since the sum of all four entries must equal one we have the following relations

$$t_{i,2} = \begin{cases} 0 & \text{if } t_{i,1} = 0 \\ 1 - t_{i,1} & \text{otherwise} \end{cases} \quad (5.1)$$

$$t_{i,3} = \begin{cases} 0 & \text{if } t_{i,0} = 0 \\ 1 - t_{i,0} & \text{otherwise} \end{cases} \quad (5.2)$$

To justify the row reduction note that the binary-indexed rows in $T^{(r)}$ correspond to correct input combinations. Hence, in order to derive the missing rows in $T$ we have to establish the relation between the output for an incorrect input combination and the output for a correct input combination. For a given input combination $i = (i_1, i_2, \cdots, i_k, \cdots, i_n)$ the output $Y$ is either sensitive to the input $X_k$ or not. The output is said to be sensitive to the input $X_k$ if a change in $X_k$ alone (from 0 to 1 or vice versa) causes a change in $Y$. Clearly, this sensitivity depends upon the values of the other input signals. If $Y$ is insensitive to $X_k$ then the correctness of $X_k$ does not influence the correctness of $Y$. Hence, in this case, replacing a correct signal $i_k \in \{0, 1\}$ by an incorrect signal $i_k + 2$ will result in an identical row in $T$. On the other hand, if $Y$ is sensitive to $X_k$ then an incorrect input signal will cause an incorrect output signal. To see the relation between row $\hat{i} = (i_1, i_2, \cdots, i_k + 2, \cdots, i_n)$ and row $i$ in $T$ consider a case in which $i_k = 0$ and the correct output is $Y = 0$. The entry $t_{i,0} = \Pr\{Y = 0 \mid X = (i_1, \cdots, i_k = 0, \cdots, i_n)\}$ depends upon the internal faults in the system. When the correct signal $i_k = 0$ is replaced by an incorrect signal $i_k = 2$ no change occurs in the internal faults and the resulting output will be an incorrect zero with the same probability, i.e., $t_{\hat{i},2} = \Pr\{Y = 2 \mid X = i_1, \cdots, i_k = 2, \cdots, i_n)\} = \Pr\{Y = 0 \mid X = (i_1, \cdots, i_k = 0, \cdots, i_n)\}$. Similarly, it can be shown that $t_{\hat{i},0} = t_{i,2}$, $t_{\hat{i},3} = t_{i,1}$ and $t_{\hat{i},1} = t_{i,3}$. In general, if $Y$ is sensitive to $X_k$ then row $\hat{i}$ is a (2, 3, 0, 1) permutation of row $i$. Consequently, every missing row in $T$ is either identical to some binary-indexed row in $T^{(r)}$ or is a (2, 3, 0, 1) permutation of such a row depending on the sensitivity of the output to the input signals. The testing of the sensitivity is simplified by the following procedure. It generates the missing rows recursively so that a check of the sensitivity to just one input signal at a time is needed.

*Procedure 5.1:* (row expansion).

*Step 1:* Set $d = 1$.

*Step 2:* Generate row $\hat{i} = (i_1, i_2, \cdots, i_k + 2, \cdots, i_n)$ where $i_k \in \{0, 1\}$ and $\hat{i}$ contains exactly $d$ nonbinary elements, from row $i = (i_1, \cdots, i_k, \cdots, i_n)$ with $d - 1$ nonbinary elements. Row $\hat{i}$ is identical to row $i$ if $Y$ is insensitive to $X_k$ and is a (2, 3, 0, 1) permutation of row $i$ if $Y$ is sensitive to $X_k$.

*Step 3:* Set $d = d + 1$ and repeat Step 2 until $d = n$.

In Step 2 we check the sensitivity of the output to the input $X_k$ only. The output $Y$ is sensitive to $X_k$ for a given input combination if a change from 0 to 1 in $X_k$ causes a change in $Y$. This check is done by comparing the row with the index $(i_1, \cdots, i_k = 0, \cdots, i_n)$ to the row with the index $(i_1, \cdots, i_k = 1, \cdots, i_n)$. If the zero entries in these two rows are in the same positions then $Y$ is insensitive to $X_k$ and vice versa. Clearly, these two rows are already known and may be compared since their indices contain just $d - 1$ nonbinary elements.

In summary, the reduced matrix $T^{(r)}$ can be expanded to the original matrix $T$ by first applying (5.1) and (5.2) for column expansion and then Procedure 5.1 for row expansion. Carrying out the column expansion is straightforward while Procedure 5.1 is slightly more complicated. However, this row expansion is not needed when performing the star operation since, by Definition 4.1, in order to form a binary-indexed row in the star product $A \circledast B$ only binary-indexed rows in $A$ and $B$ are needed. Using reduced RTM's for the standard modules decreases considerably the computational complexity of the proposed method.

## VI. Conclusions

A new approach to the evaluation of the reliability of digital systems has been presented. A reliability transfer matrix of a digital system has been defined and a procedure for its evaluation was developed. This procedure is especially efficient for systems consisting of several identical subsystems, e.g., cellular arrays and NMR systems. The extension of this approach to sequential digital systems will be presented in a subsequent paper.

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