FAULT TOLERANT SYSTEMS

http://www.ecs.umass.edu/ece/koren/FaultTolerantSystems

Part 7 - Coding 2

Chapter 3 – Information Redundancy

Cyclic Codes

- Cyclic codes are often non-separable although separable cyclic codes exist
- Encoding consists of multiplying (modulo-2) the data word by a constant number
- The coded word is the product
- Decoding is dividing by the same constant - if the remainder is non-zero, an error has occurred
- Cyclic codes are widely used in data storage and communication
- Called cyclic since - if $a_{n-1}, a_{n-2}, \cdots, a_0$ is a codeword, so is its cyclic shift $a_0, a_{n-1}, a_{n-2}, \cdots, a_1$
- Example: A 5-bit cyclic code

$$\{00000, 00011, 00110, 01100, 11000, 10001, 00101, 01010, 10100, 01001, 10010, 01111, 11110, 11101, 11011, 10111\}$$

Cyclic Codes - Theory

- k - number of bits of data that are encoded
- Encoded word of length n bits - obtained by multiplying the given k data bits by a number that is n-k+1 bits long
- The multiplier is represented as a polynomial - the generator polynomial
- 1s and 0s in the n-k+1-bit multiplier are treated as coefficients of an (n-k) -degree polynomial
- Example: multiplier is 11001 - generator polynomial is

$$G(X) = 1 \cdot X^0 + 0 \cdot X^1 + 0 \cdot X^2 + 1 \cdot X^3 + 1 \cdot X^4$$

$$= 1 + X^3 + X^4$$

An (n,k) Cyclic Code

- Using a generator polynomial of degree n-k and total number of encoded bits n
- An (n,k) cyclic code can detect all single errors and all runs of adjacent bit errors no longer than n-k
- Useful in applications like wireless communication- channels are frequently noisy and have bursts of interference resulting in runs of adjacent bit errors
- For a polynomial to be a generator polynomial for an (n,k) cyclic code it must be a factor of $X^n - 1$
- $1 + X^3 + X^4$ is a factor of $X^{15} - 1 \Rightarrow$ (15,11) code

$$X^{15} - 1 = (X+1)(X^2+X+1)(X^4+X+1)(X^4+X^3+1)(X^4+X^3+X^2+X+1)$$

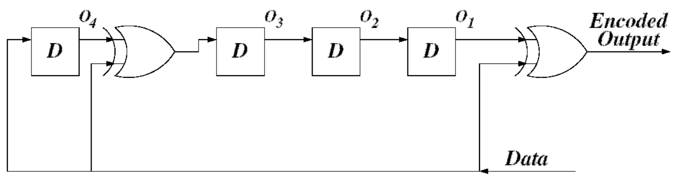
- For the 5-bit code, (X+1) - generator polynomial

$$X^5 - 1 = (X+1)(X^4+X^3+X^2+X+1)$$

- Multiply {0000, …,1111} by (X+1) to obtain all codewords of the (5,4) code

Hardware Implementation

- Multiplication can be implemented by **shift registers** and **exclusive-or** gates
- **Example:** generator polynomial $1+X^3+X^4$ (corresponding to the multiplier **11001**
- Encoding circuit:

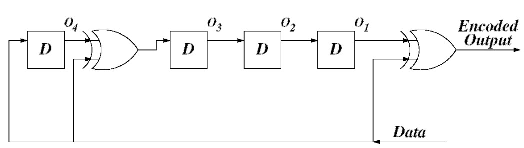


- Square boxes are delay elements which hold their input for one clock cycle

Modulo-2 Multiplication

- The **5th** bit of the product is the **modulo-2** sum of the corresponding bits of the multiplicand shifted **0** times, **3** times, and **4** times
- If multiplicand is fed in serially - we add the shifted multiplicand
- Shifting done by delay elements
- This cyclic code is **not separable** - data and check bits within **110000100011101** are not separable

```
      10001100101
  ×         11001
  ---------------
      10001100101
     00000000000
    00000000000
   10001100101
  10001100101
  ---------------
  110000100011101
```

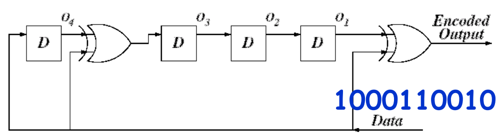


Operation of Encoding Circuit

shift clock	input data	O_4	i_3	O_3, O_2, O_1	encoded output
1	1	0	1	000	1
2	0	1	1	100	0
3	1	0	1	110	1
4	0	1	1	111	1
5	0	0	0	111	1
6	1	0	1	011	0
7	1	1	0	101	0
8	0	1	1	010	0
9	0	0	0	101	1
10	0	0	0	010	0
11	1	0	1	001	0
12	0	1	1	100	0
13	0	0	0	110	0
14	0	0	0	011	1
15	0	0	0	001	1

110000100011101

- i_3 is the input to the O_3 delay element



10001100101

Decoding Through Division by Generator Polynomial

```
110000100011101 : 11001 = 10001100101
11001
 10100
 11001
  11010
  11001
   11111
   11001
    11001
    11001
    00000
```

```
110000100111101 : 11001 = 10001100110
11001
 10100
 11001
  11011
  11001
   10111
   11001
    11100
    11001
     01011
```

- Division by 11001
- Subtraction modulo-2 - identical to addition
- Zero remainder indicates no error detected
- If a single error occurs - 110000100111101 - non-zero remainder

Three Bit Errors

```
110000111010101 : 11001 = 10001101101
11001
-----
 10111
 11001
 -----
  11100
  11001
  -----
    10110
    11001
    -----
     11111
     11001
     -----
       11001
       11001
       -----
       00000
```

```
110000011011101 : 11001 = 10001110011
11001
-----
 10011
 11001
 -----
  10100
  11001
  -----
   11011
   11001
   -----
      10110
      11001
      -----
       11111
       11001
       -----
       00110
```

- 110000111010101 instead of 110000010001110 1 - zero remainder - errors not detected (erroneously)
- If errors are adjacent - 110000011011101 - non-zero remainder indicates an error

Implementing a Divider Circuit

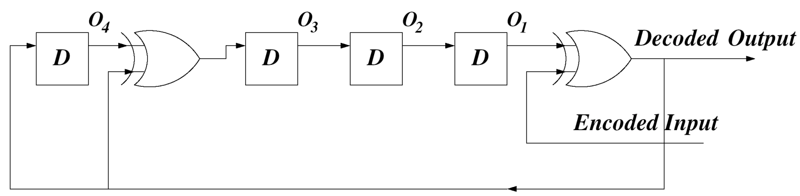
- Division can be done by multiplication in the feedback loop
- Notation: Encoded word - polynomial E(X), generator polynomial - G(X), original data word - polynomial D(X)
- If no bit errors exist - we will receive E(X) and calculate D(X) by D(X)=E(X)/G(X) - remainder will be zero
- Example: $E(X) = D(X) \cdot G(X) = D(X)(1+X^3+X^4)$

$$= D(X) + D(X)(X^3+X^4)$$

$$D(X) = E(X) - D(X)(X^3+X^4) = E(X) + D(X)(X^3+X^4)$$

* (since addition = subtraction in mod-2 arithmetic)

Divider



- Feedback circuit for division

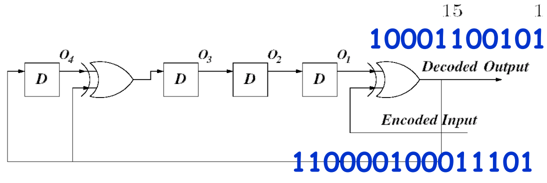
$$D(X) = E(X) + D(X)(X^3 + X^4)$$

- Start with all delay elements holding **0** and produce first the **seven** quotient bits (data bits), and then the **four** remainder bits
- If remainder bits are **nonzero**, an error has occurred

Operation of Divider

- i_3 - input to the O_3 delay element= $i_4 \oplus O_4$

shift clock	encoded input	i_4	O_4	i_3	O_3, O_2, O_1	decoded output
1	1	1	0	1	000	1
2	0	0	1	1	100	0
3	1	1	0	1	110	1
4	1	0	1	1	111	0
5	1	0	0	0	111	0
6	0	1	0	1	011	1
7	0	1	1	0	101	1
8	0	0	1	1	010	0
9	1	0	0	0	101	0
10	0	0	0	0	010	0
11	0	1	0	1	001	1
12	0	0	1	1	100	0
13	0	0	0	0	110	0
14	1	0	0	0	011	0
15	1	0	0	0	001	0



- Any error in received sequence **E(X)** will result in a **non-zero** remainder

Detecting Bursty Errors

- Many applications need to make sure that all burst errors of length 16 bits or less will be detected
- Cyclic codes of the type (16+k,k) are used
- The generating polynomial should be selected to allow a large number of data bits (use same circuit for different sizes of data blocks)
- Most commonly used :
  - CRC-16 (16-bit Cyclic Redundancy Code)

$$G(X)=X^{16}+X^{15}+X^{2}+1$$

CRC-CCITT

$$G(X)=X^{16}+X^{12}+X^{5}+1$$

Both divide $X^{n}-1 \quad for \quad 2^{15}-1=32{,}767$ bits

Separable Cyclic Codes

- Allow use of data before decoding completes
- Data word $D(X)=d_{k-1}X^{k-1}+d_{k-2}X^{k-2}+\cdots+d_{0}$
- Append (n-k) zeroes to D(X) to obtain $\overline{D}(X)=$ $d_{k-1}X^{n-1}+d_{k-2}X^{n-2}+\cdots+d_{0}X^{n-k}$
- Divide by G(X): $\overline{D}(X)=Q(X)G(X)+R(X)$, degree of R(X) < n-k
- Codeword $C(X)=\overline{D}(X)-R(X)$ has G(X) as a factor
- Divide C(X) by G(X) - if non-zero $\Rightarrow$ error
- In C(X) : first k bits data, last n-k check bits
- Example: (5,4) code with G(X)=X+1: for data 0110 we get $\overline{D}(X)=X^{3}+X^{2}=(X+1)X^{2}+0$
  - Codeword 01100
  - Same codewords generated but different correspondence

Arithmetic Codes

- Codes that are preserved under a set of arithmetic operations
- Enable detection of errors occurring during execution of arithmetic operations
- Error detection can be attained by duplicating the arithmetic processor - too costly
- A code is preserved under an arithmetic operation $*$ if for any two operands X and Y and the corresponding encoded entities X' and Y' there is an operation $\otimes$ satisfying $X' \otimes Y' = (X * Y)'$
  - The result of $\otimes$ when applied to the encoded X' and Y' will yield the same as encoding the outcome of the original operation $*$ to the original operands X and Y

Error Detection

- Arithmetic codes should be able to detect all single-bit errors
- A single bit error in an operand or an intermediate result may cause a multiple-bit error in the final result
- Example - when adding two binary numbers, if stage i of the adder is faulty, all the remaining n-i higher order digits may be erroneous

Non-Separable Arithmetic Codes

- Simplest - AN-codes - formed by multiplying the operands by a constant *A*
- X'=A•X and the operations * and ⊗ are identical for add/subtract
- Example - A=3
  - Each operand is multiplied by 3 (obtained as 2X+X)
  - The result of the operation is checked to see whether it is an integer multiple of 3
- All error magnitudes that are multiples of *A* will not be detected

AN Codes

- *A* should not be a power of the radix 2
- An odd *A* is best - it will detect every single bit fault - such an error has a magnitude of 2^i
- A=3 - least expensive AN-code that enables detection of all single bit errors
- Example - the number $0110_2 = 6_{10}$
- Representation in the AN-code with A=3 is $010010_2 = 18_{10}$
- A fault in bit position 2^3 may give the erroneous result $011010_2 = 26_{10}$
- The error is easily detectable - 26 is not a multiple of 3

Separable Arithmetic Codes

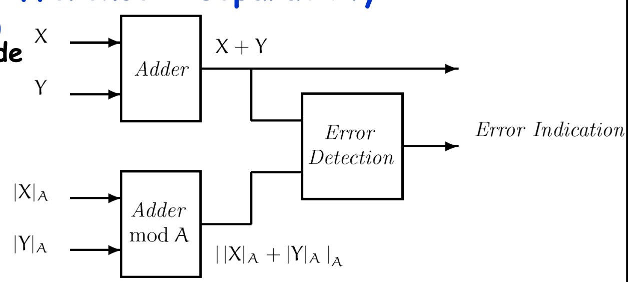
- Simplest - residue code and inverse residue code
- We attach a separate check symbol $C(X)$ to every operand X
- For the residue code, $C(X)=X \bmod A = |X|_A$
  - A is called the check modulus
- For the inverse residue code, $C(X)=A-(X \bmod A)$
- For both codes $C(X) \otimes C(Y) = C(X * Y)$
  - $\otimes$ equals $*$ - either addition or multiplication
- $|X+Y|_A = ||X|_A + |Y|_A|_A$
- $|X \bullet Y|_A = ||X|_A \bullet |Y|_A|_A$
- Division: $X-S=Q \bullet D$ - X is the dividend, D is the divisor, Q is the quotient, S is the remainder
- The check is: $||X|_A - |S|_A|_A = ||Q|_A \bullet |D|_A|_A$

Examples

- $A=3$, $X=7$, $Y=5$ - the residues are:
  $|X|_3 = 1$ and $|Y|_3 = 2$; $|7+5|_3 = 0 = ||7|_3 + |5|_3|_3 = |1+2|_3 = 0$
- $|7 \bullet 5|_3 = 2 = ||7|_3 \bullet |5|_3|_3 = |1 \bullet 2|_3 = 2$
- $A=3$, $X=7$ and $D=5$ - $Q=1$ and $S=2$ -
  the residue check is:
  $||7|_3 - |2|_3|_3 = ||5|_3 \bullet |1|_3|_3 = 2$
- Subtraction is done by adding the complement to the modulus:
  $|1-2|_3 = |1+|3-2|_3|_3 = |1+1|_3 = 2$

Residue mod A vs. AN Code

- Same undetectable errors
  - Example: A=3 - only errors that modify the result by a multiple of 3 will not be detected
  - single-bit errors are always detectable
- Same checking algorithm
  - Compute the residue modulo A of the result
- Same increase in word length - $\lceil \log_2 A \rceil$
- Most important difference - separability
  - The unit for C(X) in the residue code is separable
  - Single unit for the AN code



Low Cost Arithmetic Codes

- AN and residue codes with A=3 - simplest examples of a class of arithmetic codes with $A = 2^a - 1$ (a integer)
- Simplifies the calculation of remainder when dividing by A (checking algorithm)
- Calculating the remainder is simple since
- $| Z_i r^i |_{r-1} = | Z_i |_{r-1}$; $r = 2^a$
- Allows the use of modulo-($2^a - 1$) summation of the groups of size a bits that compose the number (each group has a value $0 \le Z_i \le 2^a - 1$)

Example

- Remainder when dividing X=11110101011 by $A=7=2^3-1$
  - Partition X into groups of size 3, starting with the least significant bit
  - This yields $X=(Z_3,Z_2,Z_1,Z_0)$=(11,110,101,011)
  - Add these groups modulo 7: "cast out" 7's and add the end-around-carry when necessary
  - Weight of carry-out is 8 ; $|8|_7=1$:
  - Add end-around-carry
  - Residue mod 7 of X is 3
  - Correct remainder of $X=1963_{10}$ divided by 7

$$|Z_i r^i|_{r-1} = |Z_i|_{r-1} \qquad r=2^a$$

	11	z_3	
+	110	z_2	
1	001		
+	1	end-around carry	
	010		
+	101	z_1	
	111		
+	011	z_0	
1	010		
	1	end-around carry	
+	011		

Arithmetic Codes with Signed Operands

- Code must be complementable with respect to R
  - $R=2^n$ (two's complement)
  - Or $R=2^n-1$ (one's complement)
  - n - number of bits in the encoded operand
- For the AN code, R-AX must be divisible by A - A must be a factor of R
- If we insist on an odd A - $R=2^n$ is excluded
- Odd A - only one's complement can be used - A must be a factor of 2^n-1

Example - AN code

- n=4, $R=2^n-1=15$ for one's complement
  - divisible by A for the AN code with A=3
- X=0110 is represented by 3X=010010
  - one's complement is 101101 = 45_{10}
  - divisible by 3
- The two's complement of 3X is 101110=46_{10}
  - not divisible by 3
- If n=5 - one's complement - R=31
  - not divisible by A=3
- X=00110 - represented by 3X=0010010
  - one's complement is 1101101=109_{10}
  - not divisible by 3

Residue Code with Signed Operands

- A-$|X|_A$ =$|R-X|_A$ must be satisfied
- R must be an integer multiple of A - again allowing only one's complement arithmetic
- Modifying the procedure so that two's complement (with $R=2^n$) can also be used -

  $$|2^n - X|_A = |2^n - 1 - X + 1|_A = |2^n - 1 - X|_A + |1|_A$$

  * Need to add a correction term $|1|_A$ to the residue code when forming the two's complement
  * A must still be a factor of $2^n - 1$

Example - Residue Code

- A=7, n=6, $R = 2^6 = 64$ for two's complement - R-1=63 is divisible by 7
- $001010_2 = 10_{10}$ has the residue 3 mod 7
- The two's complement of 001010 is 110110
- The complement of $|3|_7$ is $|4|_7$ and adding the correction term $|1|_7$ yields 5 - the correct residue mod 7 of $110110 = 54_{10}$
- Similar correction needed when adding two's complement operands and a carry-out (of weight 2^n) is generated and discarded
- To compensate, subtract $|2^n|_A$ from the residue check
- Since A is a factor of $2^n - 1$, $|2^n|_A = |1|_A$

Interdependence Between Main and Check Units

- In this two's complement addition - a carry-out is generated and discarded

$$
\begin{array}{rl}
 & 110110 = X \\
+ & 001101 = Y \\
\hline
1 & 000011
\end{array}
\qquad
\begin{array}{rll}
 & 101 = |X|_7 & \\
+ & 110 = |Y|_7 & \\
\hline
1 & 011 & \\
 & \quad 1 & \text{end-around carry} \\
\hline
 & 100 & \\
- & \quad 1 & \text{correction term} \\
\hline
 & 011 &
\end{array}
$$

- Interdependence between the main and check units
- An error in the main unit may propagate to the check unit and the effect of the fault is masked
- A single-bit error is always detectable

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Bi-Residue Code

- Error correction can be achieved by using two or more residue checks
- Simplest case - bi-residue code
- Consists of two residue checks A_1 *and* A_2
- $A_1 = 2^a - 1$ and $A_2 = 2^b - 1$ are two low-cost residue checks with n=l.c.m.(a,b)
  - n is the number of bits in the operands
- Any single-bit error can be corrected

Part.7 .29