§10.4.1. Power Spectral Density

o energy signal & energy spectral density

· Def.

A signal (s(t)) as a time function is called an energy signal or a signal of finite energy if

$$E = \int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$$

· Def.

If an energy signal s(t) has its Fourier transform S(f), then the energy spectral density of s(t) is defined as

$$E(f) = |S(f)|^2$$

· Remark

By Parseval's relation

$$\int_{-\infty}^{\infty} x(t)\, y(t)^* dt = \int_{-\infty}^{\infty} X(f)\, Y(f)^* df$$

Let $x(t) = y(t) = s(t)$. Then

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |S(f)|^2 df$$

$$= \int_{-\infty}^{\infty} E(f)\, df$$

o Power signal & power spectral density

· Def.

Given a signal $s(t)$, the average power during the period $[a,b]$ is defined as

$$\frac{\int_a^b |s(t)|^2 dt}{b-a} \quad \left(= \frac{\text{energy}}{\text{time}} \right)$$

· Def.

Given a signal $s(t)$, the instantaneous power is defined as

$$|s(t)|^2 \quad \left(= \lim_{a \to x} \frac{\int_{\min(a,x)}^{\max(a,x)} |s(t)|^2 dt}{|x-a|} \right)$$

· Def.

A signal $s(t)$ is called a power signal or a signal of finite power if

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |s(t)|^2 dt < \infty$$

$$\left(= \lim_{\alpha \to \infty} \frac{1}{\alpha T} \int_{\langle \alpha T \rangle} |s(t)|^2 dt \right)$$

· Def.

Define $S_T(t) \triangleq s(t) \{ u(t+T) - u(t-T) \}$ and define $S_T(f) \triangleq \mathcal{F}\{S_T(t)\}$. Then,

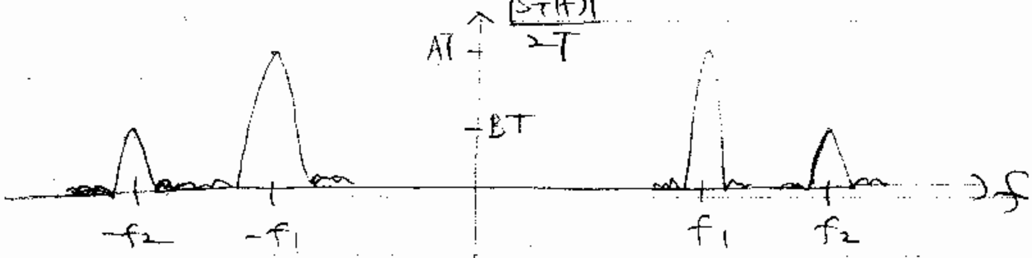
$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} |S_T(t)|^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} |S_T(f)|^2\, df \\
&\overset{?}{=} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} \frac{|S_T(f)|^2}{2T}}_{\text{power spectral density}}\, df
\end{aligned}
$$

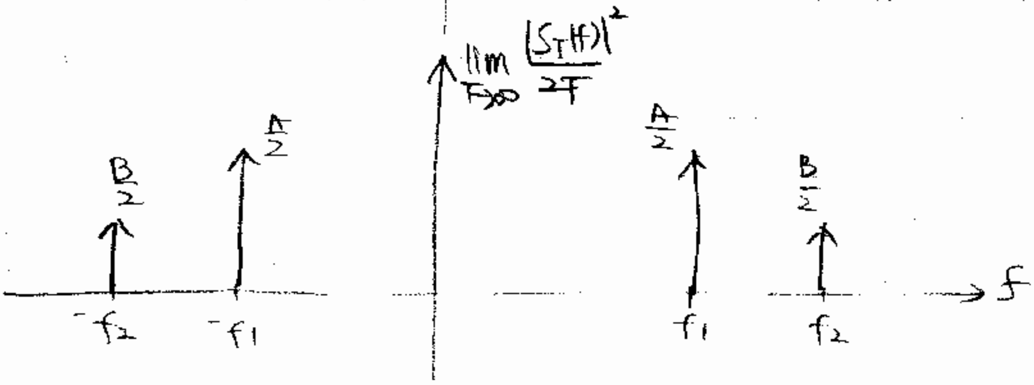
ex1

$$S(t) = \sqrt{2A}\cos 2\pi f_1 t + \sqrt{2B}\cos 2\pi f_2 t$$

$$S_T(t) = S(t)\,[u(t+T) - u(t-T)]$$

$$S_T(f) = \Big[\sqrt{\tfrac{A}{2}}\,\{\delta(f-f_1) + \delta(f+f_1)\} + \sqrt{\tfrac{B}{2}}\,\{\delta(f-f_2) + \delta(f+f_2)\}\Big] * 2T\,\frac{\sin 2\pi f T}{2\pi f T}$$





Remark

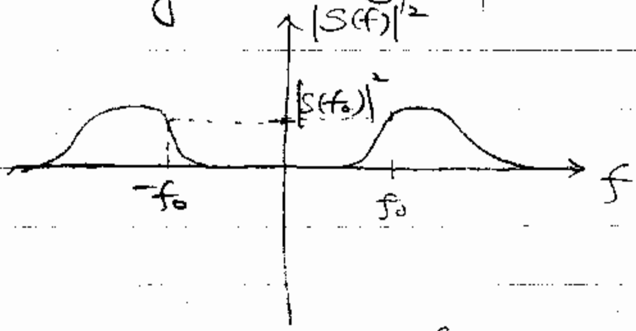
If $\quad S(t) = \sum_{i=1}^{N} \sqrt{2P_i}\cos(2\pi f_i t + \theta_i)$

then $\quad \lim_{T\to\infty} \frac{|S_T(f)|^2}{2T} = \sum_{i=1}^{N} \left\{ \frac{A}{2}\big[\delta(f-f_i) + \delta(f+f_i)\big] \right\}$.

∘ How to measure spectral densities?

• measuring an energy spectral density

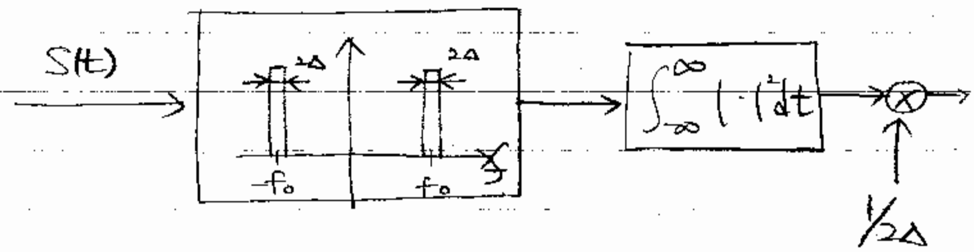
$s(t)$: real-valued



$$\underset{\text{e.s.d at } f=f_0}{|S(f_0)|^2} \simeq \frac{1}{2\Delta} \underbrace{\int_{f_0-\Delta}^{f_0+\Delta} |S(f)|^2 df}_{\text{energy in } f\in[f_0-\Delta,\, f_0+\Delta]} \quad w/ \quad \Delta \approx 0$$

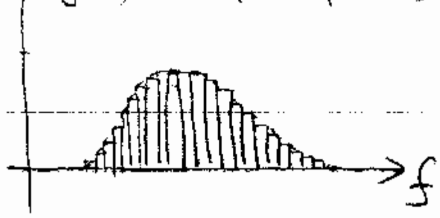
∴

step1



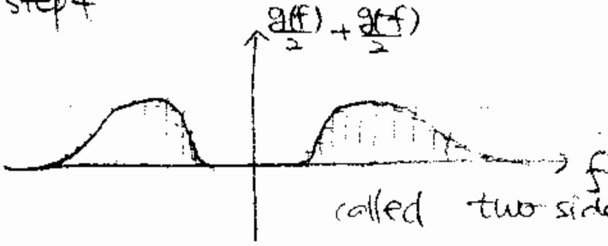
step2 Change $f_0 \in (0, \infty)$. Repeat step 1.

step3 $g(f) = 2|S(f)|^2 u(f)$



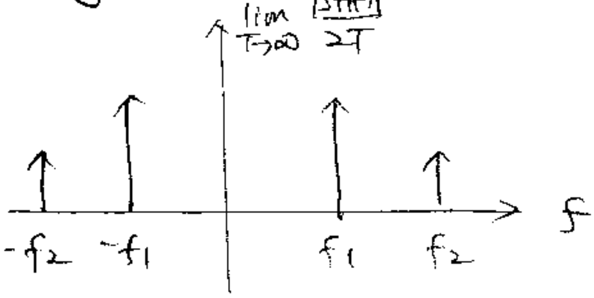
called one-sided energy spectral density.

step4



called two sided energy spectral density.

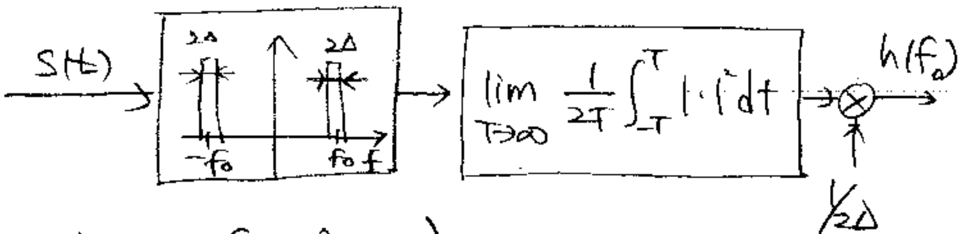
- measuring a power spectral density



$$\frac{1}{2\Delta}\int_{f_0-\Delta}^{f_0+\Delta} \lim_{T\to\infty} \frac{|S_T(f)|^2}{2T}\, df \simeq \text{p.s.d. at } f=f_0$$

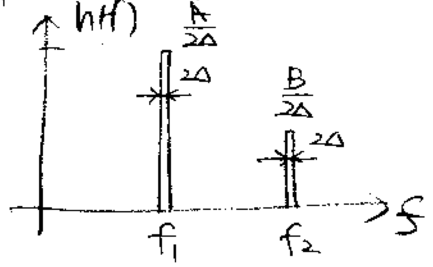
$\Delta \approx 0$

$\therefore$ step 1



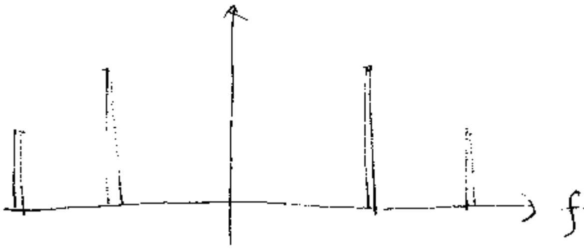
step2 Change $f_0 \in [0, \infty)$

step3 Plot $h(f)$



called one-sided PSD.

step4 Plot $\frac{h(f)+h(-f)}{2}$



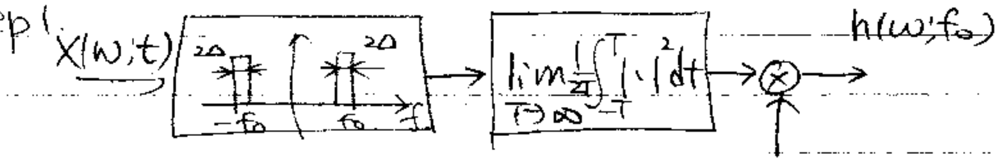
called two-sided PSD.

- PSD of a random process
  - Example

$$X(t) = \sqrt{2P}\cos(2\pi f_c t + \Theta), \quad -\infty < t < \infty$$

where Θ is a uniform r.v. on $[0, 2\pi)$. f_c is a r.v. Each sample path (function) is a power signal w/ psd $\frac{P}{2}\{\delta(f-f_c)+\delta(f+f_c)\}$ regardless of Θ.

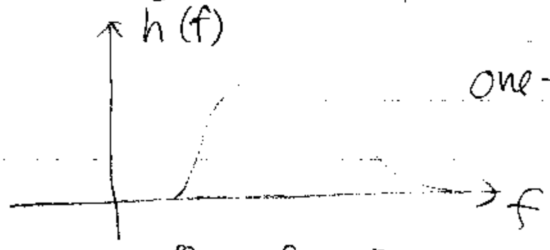
- If PSD can be defined for random processes, the definition must result in in order for the definition to be useful. More over, the PSD must be measurable by

Step 1 

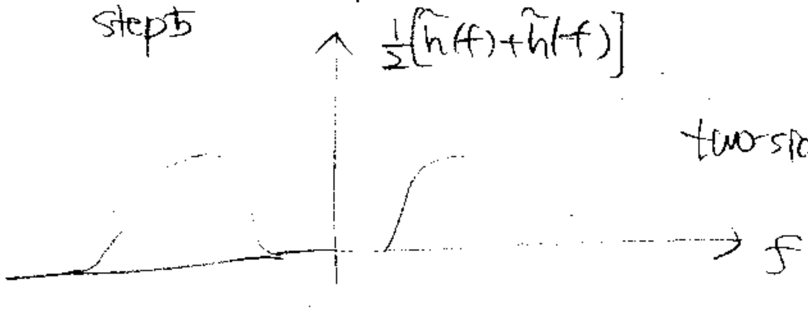
sometimes not possible. → Step 2. Repeat step 1 $\forall w \in \Omega$

Then $\tilde{h}(f_0) = E[h(f_0)]$

step 3. Repeat [step1, step2] $\forall f \in [0, \infty)$

step 4 

one-sided PSD

step 5 

two-sided PSD

- It was shown that if $\Delta \approx 0$ this procedure gives us

$$\mathcal{F}\{A[R_{XX}(t,t+\tau)]\}$$

where

$$R_{XX}(t,t+\tau) \triangleq E\{X(t)X(t+\tau)\}$$

$$A[\ \ ] \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\ \ ]\,dt \quad \leftarrow \text{time average.}$$

$$\mathcal{F}\{\ \ \} \triangleq \int_{-\infty}^{\infty}\{\ \ \}e^{-j2\pi f\tau}d\tau \quad \leftarrow \text{Fourier transform.}$$

- Def.

The PSD $S_{XX}(f)$ of a r.p. $X(t)$ is defined as

$$S_{XX}(f) \triangleq \int_{-\infty}^{\infty}\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_{XX}(t,t+\tau)\,dt\right\}e^{-j2\pi f\tau}d\tau$$

where

$$R_{XX}(t,t+\tau) \triangleq E\{X(t)X(t+\tau)\}$$

is the auto-correlation function of $X(t)$.

- Remark

(1) If $X(t)$ is WSS, then

$$R_{XX}(\tau) \xleftrightarrow{\mathcal{F}} S_{XX}(f),$$

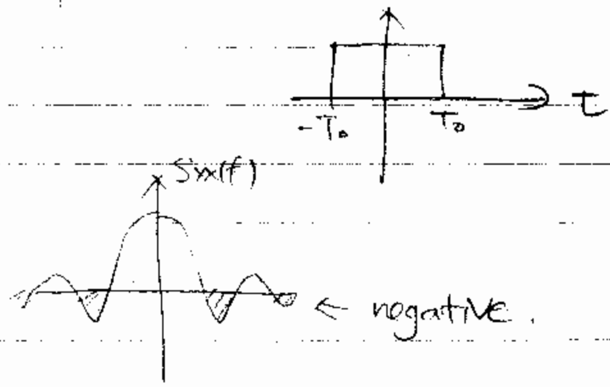
i.e., $$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)\,e^{j2\pi f\tau}df$$

(ii) $R_{XX}(0) = E\{X(t)X(t+0)\} = E\{|X(t)|^2\}$ ← average power

$$= \int_{-\infty}^{\infty} \underbrace{S_{XX}(f)}_{\text{called PSD.}}\, df$$

(iii) $S_{XX}(f) \geq 0$. Hence, $R_{XX}(\tau)$ can not be arbitrary.

Ex/ Is it possible to have $R_{XX}(\tau)$

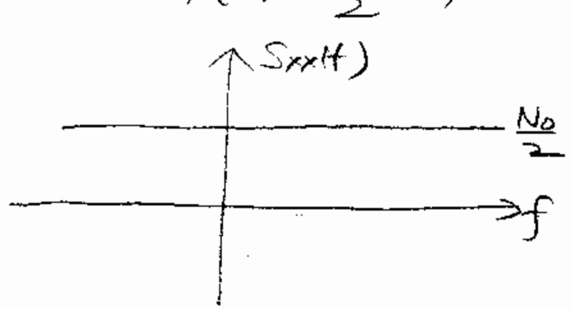


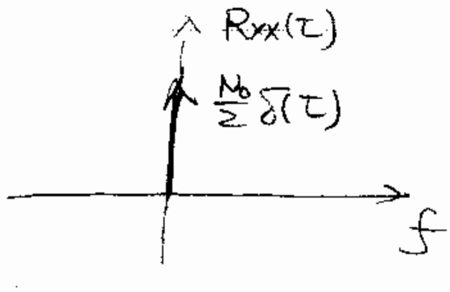
No!

§ 10.4.2 White Noise

· Def. A white noise is a zero-mean WSS random process with PSD

$$S_{XX}(f) = \frac{N_0}{2}, \quad -\infty < f < \infty$$





- Remarks

(i) $E[X(t_1)X(t_2)] = 0 \quad \forall t_1 \neq t_2$

(ii) $E[X(t)^2] = \infty \quad \forall t$

Note a white noise is an approximation of a zero-mean WSS r.p. w/

$$S_{XX}(f) = \begin{cases} \frac{N_0}{2} & |f| \leq B \\ 0 & |f| > B \end{cases}$$

for very large B.

- Def.

A white Gaussian noise is an approximation of a zero-mean WSS Gaussian r.p. w/

$$S_{XX}(f) = \begin{cases} \frac{N_0}{2} & |f| \leq B \\ 0 & |f| > B \end{cases}$$

for very large B.

- In page 15-6, step 2 may be impossible or unnecessary for some r.p.'s. We define such a class of r.p.'s.

we observe only one $\omega \in \Omega$

- Note that

$$S_{XX}(f) \triangleq \mathcal{F}\{A\{E[X(t)X(t+\tau)]\}^?\}$$

Since we can observe only one $\omega \in \Omega$, the statistical average cannot be approximated as the arithmetic average of <u>many i.i.d.</u> observations.

So what we want is

$$\mathcal{F}\left\{A\left\{E[X(t)X(t+\tau)]\right\}\right\}$$
$$= \mathcal{F}\left\{A\left[X(t)X(t+\tau)\right]\right\}$$

, i.e.,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(\omega;t)X(\omega;t+\tau)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E\left[X(t)X(t+\tau)\right]dt$$

for almost every $\omega \in \Omega$ (almost sure equality)

However, this is too much as a.s. convergence of a r. sequence is too much for most applications.

- We want to define like

"A WSS r.p $X(t)$ is ergodic in the mean if

$$\lim_{T\to\infty} E\left\{\left|\frac{1}{2T}\int_{-T}^{T} X(t)\,dt - \mu\right|^2\right\} = 0 \text{ "}$$

This resembles the convergence of a random sequence in the quadratic mean

- We want to handle such expected value of linearly or non-lineary processed random processes.
  → We need so-called "mean-square calculus."