5. The following transfer function is a lead network designed to add about 60° of phase at \( \omega_1 = 3 \text{ rad/sec} \):

\[
H(s) = \frac{s + 1}{0.1s + 1}.
\]

(a) Assume a sampling period of \( T = 0.25 \text{ sec} \), and compute and plot in the \( z \)-plane the pole and zero locations of the digital implementations of \( H(s) \) obtained using (1) Tustin’s method and (2) pole-zero mapping. For each case, compute the amount of phase lead provided by the network at \( z_1 = e^{j\omega_1 T} \)

(b) Using a log-log scale for the frequency range \( \omega = 0.1 \) to \( \omega = 100 \text{ rad/sec} \), plot the magnitude Bode plots for each of the equivalent digital systems you found in part (a), and compare with \( H(s) \). (Hint: Magnitude Bode plots are given by \( |H(z)| = |H(e^{j\omega T})| \).)

**Solution:**

(a)

\[
H(s) = \frac{s + 1}{0.1s + 1}, \quad \angle H(j\omega)|_{\omega=3} = 54.87^\circ
\]

From Matlab, \([\text{mag, phasew1}] = \text{bode([1 1],[.1 1],[.1 1],3) yields phasew1 = 54.87.}

(1) Tustin’s method, analytically :

\[
H(z) = H(s)|_{s = \frac{2}{T} - 1, z^{-1}} = \frac{(2 + T) + (T - 2)z^{-1}}{(0.2 + T) + (T - 0.2)z^{-1}}
\]

\[
= \frac{5z - 0.7778}{z + 0.1111}
\]

or, via Matlab:

\[
\text{sysC} = \text{tf([1 1],[1 1])};
\]

\[
\text{sysDTust} = \text{c2d(sysC,T,'tustin')}
\]

Phase lead at \( \omega_1 = 3 \) : \( \angle H(e^{j\omega_1 T}) = 54.90^\circ \), which is most easily obtained by Matlab

\([\text{mag, phasew1}] = \text{bode(sysDTust,3)}
\]

The pole-zero plot is:
(2) Matched pole-zero method, analytically:

\[
H(z) = \frac{Kz - e^{-1T}}{z - e^{-10T}} = \frac{4.150z - 0.7788}{z - 0.0821}
\]

\[
K = 4.150 \implies |H(z)|_{z=1} = |H(s)|_{s=0}
\]

or, alternatively via Matlab

\[
\text{sysDmpz = c2d(sysC,T,'matched')}
\]

will produce the same result.

Phase lead at \( \omega_1 = 3 \): \( \angle H(e^{j\omega_1 T}) = 47.58^\circ \) is obtained from

\[
[mag,phasew1] = \text{bode(sysDmpz,3)}.
\]

The pole-zero plot is below. Note how similar the two pole-zero plots are.

(b) The Bode plots match fairly well until the frequency approaches the half sample frequency (\( \approx 12 \text{ rad/sec} \)), at which time the
curves diverge.

<table>
<thead>
<tr>
<th>Frequency (rad/sec)</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
<tr>
<td>10^-1</td>
<td>10^1</td>
</tr>
</tbody>
</table>

6. The following transfer function is a lag network designed to introduce a gain attenuation of \(10(-20\text{dB})\) at \(\omega_1 = 3\ \text{rad/sec}:\

\[
H(s) = \frac{10s + 1}{100s + 1}.
\]

(a) Assume a sampling period of \(T = 0.25\ \text{sec}\), and compute and plot in the \(z\)-plane the pole and zero locations of the digital implementations of \(H(s)\) obtained using (1) Tustin’s method and (2) pole-zero mapping. For each case, compute the amount of gain attenuation provided by the network at \(z_1 = e^{j\omega_1 T}\).

(b) For each of the equivalent digital systems in part (a), plot the Bode magnitude curves over the frequency range \(\omega = 0.01\) to 10 rad/sec.

**Solution:**

(a) First, we'll compute the attenuation of the continuous system,

\[
H(s) = \frac{10s + 1}{100s + 1}, \quad |H(j\omega)|_{\omega=3} = 0.1001 \quad (-20 \text{ db})
\]

(1) Tustin’s method:

\[
H(z) = H(s)|_{s=\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}} = \frac{(20 + T) + (T - 20)z^{-1}}{(200 + T) + (T - 200)z^{-1}}
\]

\[
= 0.10112z - 0.97531
\]

or, use **c2d** as shown for problem 5.
Gain attenuation at $\omega_1 = 3$ : $|H(e^{j\omega_1 T})| = 0.1000 \ (\text{−20 db})$, most easily computed from: $[\text{mag,phase}]=\text{bode(sysDTust,T,3)}$.

(2) Matched pole-zero method:

$$H(z) = K \frac{z - e^{-0.1T}}{z - e^{-0.01T}} = 0.10113 \frac{z - 0.97531}{z - 0.99750}$$

$$K = 0.10113 \quad |H(z)|_{z=1} = |H(s)|_{s=0}$$

Gain attenuation at $\omega_1 = 3$ : $|H(e^{j\omega_1 T})| = 0.1001 \ (\text{−20 db})$, most easily computed from: $[\text{mag,phase}]=\text{bode(sysDmpz,T,3)}$.

In this case, the sampling rate is so fast compared to the break frequencies that both methods give essentially the same equivalent, and both have a gain attenuation of a factor of 10 at $\omega_1 = 3$ rad/sec.

(b) All three are essentially the same and indistinguishable on the plot because the range of interest is below the half sample fre-
7. Consider the linear equation $Ax = b$, where $A$ is an $n \times n$ matrix. When $b$ is given, one way of solving for $x$ is to use the discrete-time recursion

$$x(k+1) = (I + cA)x(k) - cb,$$

where $c$ is a scalar to be chosen.

(a) Show that the solution of $Ax = b$ is the equilibrium point $x^*$ of the discrete-time system. An equilibrium point $x^*$ of a discrete-time system $x(k+1) = f(x(k))$ satisfies the relation $x^* = f(x^*)$.

(b) Consider the error $e(k) = x(k) - x^*$. Write the linear equation that relates the error $e(k+1)$ to $e(k)$.

(c) Suppose $|1 + c\lambda_i(A)| < 1, i = 1, \ldots, n$, where $\lambda_i(A)$ denotes the $i$th eigenvalue of $A$. Show that starting from any initial guess $x_0$, the algorithm converges to $x^*$. [Hint: For any matrix $B$, $\lambda_i(I + B) = 1 + \lambda_i(B)$]

Solution:

(a) For an equilibrium, $x^*$, we have:

$$x^* = (I + cA)x^* - cb$$

$$\implies cAx^* = cb$$

$$\implies Ax^* = b$$