ECE344, Midterm Exam 1: Solutions
October 19, 2009

1. **Problem:** The workfunction of a material is defined as the energy required to remove an electron from the material and is usually indicated by the Greek letter χ (pron: 'khi').

   The workfunction of gold is $\chi_{Au} = 4.9$ eV, the workfunction of cesium is $\chi_{Cs} = 1.9$ eV.

   a. Calculate the maximum wavelength (in nm) of light required to remove an electron from Au via the photoelectric effect.

   b. Do the same for Cs.

   Solution: Since
   $$\lambda \; = \; \frac{c}{\nu} \; = \; \frac{2\pi\hbar c}{h\nu} \; = \; \frac{2\pi\hbar}{E} \; , \tag{1}$$
   or, expressing the energy in eV:
   $$\lambda \; = \frac{2\pi\hbar}{eE} \; , \tag{2}$$
   using $E = \chi_{Au}$ we have $\lambda = 253$ nm and using $E = \chi_{Cs}$ we get $\lambda = 653$ nm.

2. **Problem:** Electrons in a 'swarm' at thermal equilibrium have each an average kinetic energy of $(3/2)k_BT$.

   a. Calculate the value of $(3/2)k_BT$ in eV.

   b. Calculate the velocity (in cm/s) of an electron having a kinetic energy equal to the average energy $(3/2)k_BT$. This is the so-called 'thermal velocity'.

   c. Calculate its average de Broglie wavevelength in cm. This is called the 'thermal wavelength'.

   Solution: Plugging the appropriate numbers, we get $E_{th} = (3/2)k_BT = 38.77$ meV. From $E_{th} = (1/2)mv_{th}^2$ we get for the thermal velocity $\upsilon_{th} = \sqrt{2E_{th}/m} = 1.16 \times 10^7$ cm/s. Finally, from $\lambda_{th} = h/(m\upsilon_{th})$ we get for the thermal wavelength $\lambda_{th} = 6.23$ nm.

3. **Problem:** Consider a quantum well of GaAs (which is like the one-dimensional 'box' we have considered in class): Electrons are assumed to be in one dimension, assume the x axis to be along this direction, the potential is zero in the interval

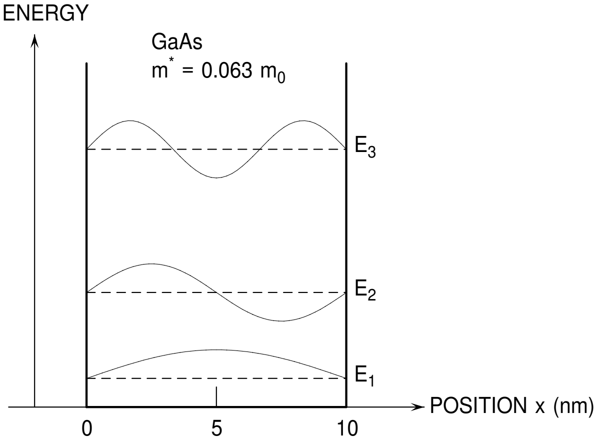

Fig. 1: Figure relative to problem 3

$(0, L)$, infinite outside, so the the wavefunction of the electrons must vanish at $x = 0$ and $x = L$.

Using the effective mass of GaAs, $m^* = 0.063 m_0$ (where $m_0 = 9.1 \times 10^{-31}$ Kg is the electron mass in vacuum), and assuming $L = 10$ nm, calculate the energy (in eV) of the three lowest-energy levels in the quantum well.

Figure 1 illustrates schematically the potential well, the position of the first 3 energy levels in the well, and the corresponding wavefunctions.

Solution: The energy levels E_n of the particle in the box are given by:

$$E_n = \frac{\hbar^2}{2m^*} \left(\frac{n\pi}{L} \right)^2 = \frac{\hbar^2 \pi^2}{2m^* L^2} n^2 \ . \tag{3}$$

Using the values for L and m^* specified by the problem, we get $E_n = 38 n^2$ meV. So, $E_1 = 38$ meV, $E_2 = 152$ meV, and $E_3 = 342$ meV.

4. **Problem:** Figure 2 shows a band-diagram for some ideal material along one particular direction in $\mathbf{k}$-space.

   Indicate the sign of the effective mass and of the velocity (along the x-axis in Fig. 2) for electrons with wavevectors $\mathbf{k}$ at each of the points labeled A, B, C, and D.

   Solution: Recall that for our 'isotropic' case:

$$m^* = \frac{\hbar^2}{d^2E/dk^2} \tag{4}$$

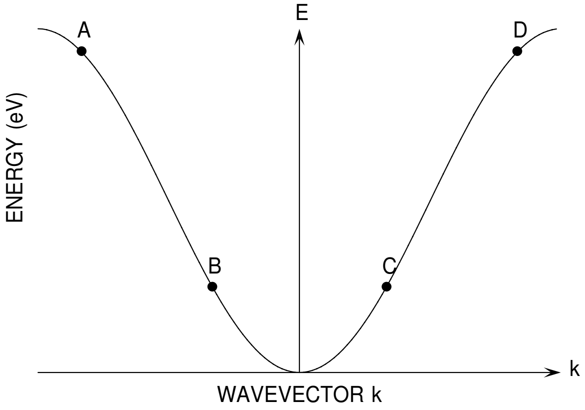

Fig. 2: Figure relative to problem 4

and

$$v = \frac{dE/dk}{\hbar}\ , \tag{5}$$

so that positive (negative) curvature implies a positive (negative) effective mass, a positive (negative) slope a positive (negative) x-component of the velocity. Thus:
At point A: $m^* < 0, \quad v < 0$
At point B: $m^* > 0, \quad v < 0$
At point C: $m^* > 0, \quad v > 0$
At point D: $m^* < 0, \quad v > 0$.
Some confusion might have been caused by associating the momentum $p = m^* v$ with $\hbar k$. This is correct only near the bottom of the band (positive effective mass) at $k = 0$, while near the top of the band (negative effective mass) the momentum is $p \approx \hbar(k - k_0)$, where k_0 is the location of the band maximum.

5. **Problem:** Calculate the probability that a single state at the bottom of the conduction band will be occupied at 300 K in Si, Ge, and GaAs. Assume that the Fermi level is at midgap (that is: $E_F = E_i = E_{mg}$ and use for the values of the gap $E_g = 1.12$ eV (for Si), 0.64 eV (for Ge), 1.42 eV (for GaAs).
   Solution: The probability of a state at the bottom of the conduction band to be occupied is given by the Fermi-Dirac distribution at $E - E_F = E_{mg}$. Thus we must evaluate

$$P = \left[1 + \exp\left(\frac{E_{mg}}{k_B T}\right)\right]^{-1}$$

for the three cases given, obtaining 3.91×10^{-10} for Si, 4.21×10^{-6} for Ge, and 1.18×10^{-12} for GaAs.

6. **Bonus problem.** Consider the conduction band of a crystal whose energy-vs-wavevector relation $E(k)$ along some direction in $\mathbf{k}$-space is given by the following expression:

$$E(k) \;=\; E_0 \;+\; E_1 \;\cos[\alpha(k-k_0)] \;.$$

Note that this band has a minimum at $k = k_0$, pretty much like Si along the line from the symmetry point Γ to the symmetry point X.

Calculate the electron effective mass near the minimum of the band in terms of the parameters of the problem, E_0, E_1, k_0 and α. (Not all of them will necessarily appear in the final answer).

Solution: From Eq. (4) and from the fact that

$$\frac{dE(k)}{dk} \;=\; -\;\alpha\; E_1\;\sin[\alpha(k-k_0)] \;,$$

$$\frac{d^2E(k)}{dk^2} \;=\; -\;\alpha^2\; E_1\;\cos[\alpha(k-k_0)] \;,$$

which is $-E_1\alpha^2$ at $k - k_0$, we have $m^* = -\hbar/(\alpha^2 E_1)$.