INT. J. CONTROL, 1998, VOL. 70, NO. 4, 603–616

Feedback linearizing control of discrete-time nonlinear systems with input constraints

MICHAEL J. KURTZ† and MICHAEL A. HENSON†

A feedback linearizing control strategy for discrete-time nonlinear systems subject to input constraints is proposed. The control system comprises: (*i*) a feedback linearizing controller; (*ii*) a constraint mapping algorithm that transforms the actual input constraints into constraints on the feedback linearized system; and (*iii*) a linear model predictive controller that regulates the resulting constrained linear system. Closed-loop stability analysis is a challenging problem because the transformed constraints are state dependent. Sufficient conditions for asymptotic stability are presented for fully and partially feedback linearizable systems. As part of the analysis, a new stability result for unconstrained discrete-time nonlinear systems which parallels a well-known continuous-time result is derived.

1. Introduction

Many nonlinear chemical processes are subject to input constraints that are encountered during routine operation. A major disadvantage of feedback linearizing control (FLC) techniques (Henson and Seborg 1997, Isidori 1989) is their lack of explicit constraint handling capabilities (Rawlings *et al.* 1994). Stability analysis is considerably more difficult than in the unconstrained case because active constraints preclude exact linearization of the closed-loop system. Feedback linearizing controllers may be detuned to avoid input constraints (Aguilar *et al.* 1996, Pappas *et al.* 1995), but this usually leads to unnecessarily poor performance (Alvarez *et al.* 1991).

While a variety of controller design and analysis techniques are available for constrained linear systems (Blanchini and Miani 1996, Rawlings and Muske 1993, Saberi *et al.* 1996), significantly fewer results have appeared for feedback linearizable nonlinear systems subject to constraints. Stability of feedback linearized systems in the presence of input constraints has been analysed (Aguilar *et al.* 1996, Alvarez *et al.* 1991, Papas *et al.* 1995). Most of these results are based on determining operating regions where the closed-loop system will evolve such that constraints are not violated. A simple modification of the basic FLC approach, which accounts for input constraints, involves applying linear anti-windup schemes to the feedback linearized system (Kendi and Doyle 1995, 1997). The disadvantage of this approach is that input constraints are not considered explicitly as part of the controller design. Instead, the controller is combined with an anti-windup compensator designed to minimize performance degradation caused by constraints.

Optimization-based design techniques such as nonlinear model predictive control (NMPC) (Alamir and Bornard 1994, Mayne and Michalska 1990, Meadows *et al.*

Received 4 March 1997. Revised 19 January 1998.
† Department of Chemical Engineering, Louisiana State University, Baton Rouge, LA 70803-7303, USA. E-mail: henson@nlc.che.lsu.edu

1995) represent an alternative approach for regulating constrained nonlinear systems. Unlike anti-windup schemes, input constraints are explicitly handled by including them as inequality constraints in the associated nonlinear program. The major difficulty associated with the NMPC approach is the solution of the nonlinear programming problem (Zhu *et al.* 1997). It is very difficult to establish verifiable conditions which guarantee the existence of a feasible solution (Achhab *et al.* 1994). Convergence of the iterative calculations required to solve the optimization problem is another potential problem. The nonlinear program is non-convex because the nonlinear model is posed as an equality constraint. Therefore, iterative algorithms may converge to a local minimum, or even diverge. Moreover, large computational effort may be required as the nonlinear programming problem must be resolved at each time step.

We have developed an alternative method for handling input constraints within the FLC framework (Kurtz and Henson 1996, 1997a, 1997b). The technique is applicable to continuous-time nonlinear systems with an equal number of manipulated inputs and controlled outputs. The basic idea is to map the actual input constraints into corresponding constraints on the discretized version of the feedback linearized system. This yields a discrete-time linear system with inputs that are subject to *state-dependent* constraints. Instead of employing a simple pole placement design (Isidori 1989), the feedback linearized system is regulated with a *linear* model predictive controller with explicit constraint handling capability (Rawlings and Muske 1993). Thus, the proposed strategy attempts to combine the benefits of the FLC and NMPC approaches. A similar approach has been proposed by Nevistic and Morari (1995).

Figures 1 and 2 (Kurtz and Henson 1997b) show the set-point tracking performance obtained when the hybrid strategy (FLC-MPC) is applied to a highly nonlinear chemical reactor model. The model describes an irreversible, first-order reaction $A \rightarrow B$ which occurs in a constant volume, continuous stirred tank reactor. The objective is to control the reactor temperature by manipulating the temperature of the coolant stream, which has a lower constraint of 280 K. The figures also show responses obtained with a conventional linear model predictive controller (LMPC) (Muske and Rawlings 1993b) and a nonlinear model predictive controller based on successive linearization of the nonlinear model (SLMPC) (Garci 1984). A detailed discussion of the reactor model and the design of the three controllers is presented elsewhere (Kurtz and Henson 1997b).

LMPC yields an oscillatory response for the positive set-point change (figure 1) and a very sluggish response for the negative set-point change (figure 2). The performance of SLMPC is acceptable for the positive change, but an unstable response is obtained for the negative change because the controller repeatedly switches between stable and unstable models. By contrast, FLC-MPC yields good closed-loop performance for both set-point changes. We also have compared (Zhu *et al.* 1997) the FLC-MPC strategy to nonlinear anti-windup (Kendi and Doyle 1995) and nonlinear model predictive control (Meadows *et al.* 1995) schemes using a polymerization reactor model. The hybrid scheme provides the best compromise between closed-loop performance and computational efficiency.

These simulation results motivate a more theoretical analysis of the FLC-MPC control strategy. The hybrid method involves discretization of the linearized model for LMPC design. It is well known that discretization represents an obstruction to exact feedback linearization (Grizzle and Kokotovic 1988). Consequently, the FLC-

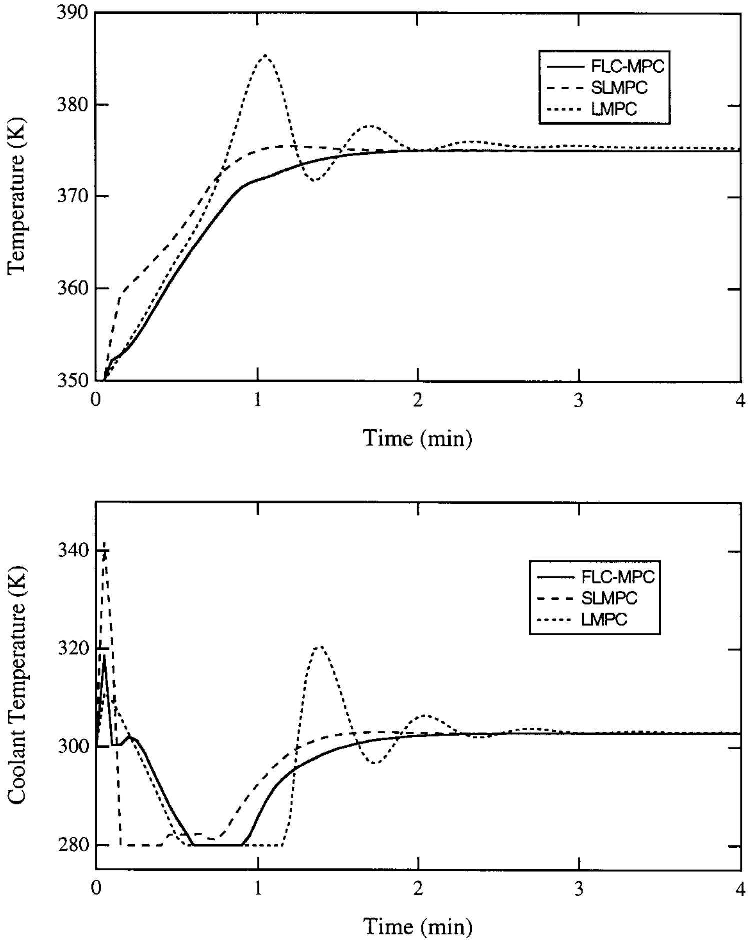

Figure 1. Positive setpoint change for chemical reactor example.

MPC technique is difficult to analyse when applied to a continuous-time nonlinear system. Some stability results based on the contraction mapping theorem have been presented for this case (Nevistic and Re 1994). However, the applicability of these results is severely limited by the assumption that the overall closed-loop operator is a contraction. In this paper, we develop and analyse a hybrid FLC-MPC strategy for *discrete-time* nonlinear systems with input constraints.

The remainder of the paper is organized as follows. In section 2, feedback linearization of unconstrained nonlinear systems is reviewed. The hybrid FLC-MPC technique for constrained nonlinear systems is presented in section 3. In section 4, a detailed stability analysis of the proposed technique is conducted. Finally, a summary and conclusions are presented in section 5.

2. Unconstrained nonlinear systems

First, we review the input–output linearization method for discrete-time nonlinear systems without input constraints (Lee *et al.* 1987, Monaco and

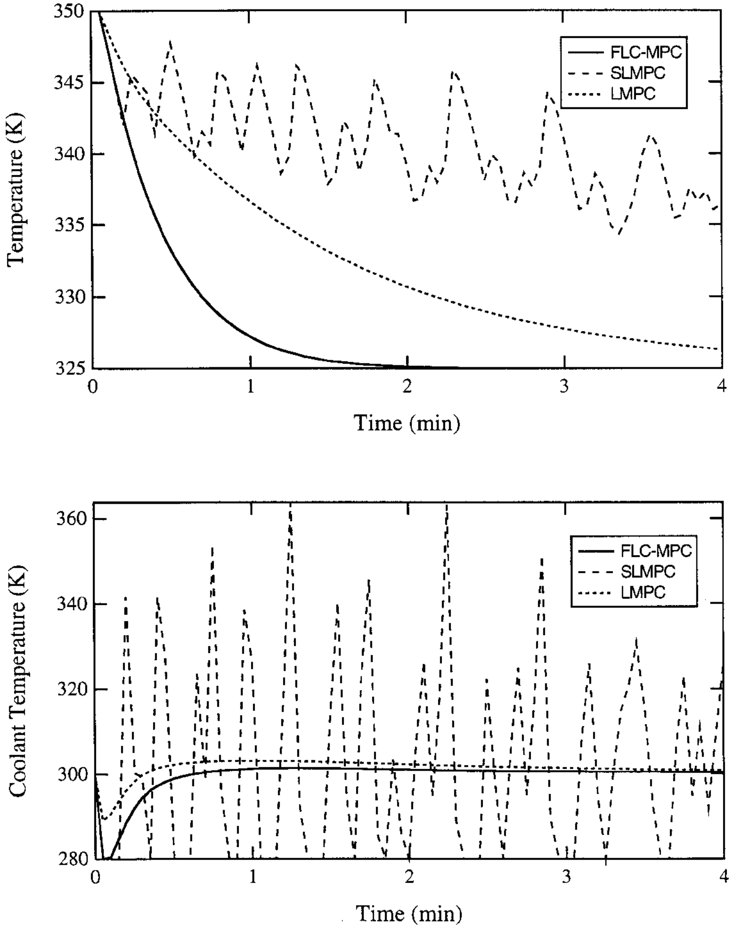

Figure 2. Negative setpoint change for chemical reactor example.

Normand-Cyrot 1987, Nijmeijer and van der Schaft 1990). Controller design is based on the state-space model

$$\begin{aligned} x(k+1) &= f\left[x(k), u(k)\right] \\ y(k) &= h\left[x(k)\right] \end{aligned} \tag{1}$$

where x is an n-dimensional state vector, u is a scalar input variable, and y is a scalar output variable. We assume without loss of generality that $f(0,0) = h(0) = 0$. The relative degree (r) characterizes the effective time delay between the input (u) and the output (y). The value of the output at time $k+j, 1 \le j \le r$, can be written as

$$y(k+j) = h \circ f^{(j)}\left[x(k), u(k)\right] \tag{2}$$

where '$\circ$' is the composition operator and the function $f^{(j)}$ is defined recursively as

$$\left.\begin{aligned} f^{(1)}\left[x(k), u(k)\right] &= f\left[x(k), u(k)\right] \\ f^{(j)}\left[x(k), u(k)\right] &= f^{(j-1)} \circ f\left[x(k), u(k)\right] \end{aligned}\right\} \tag{3}$$

The relative degree is formally defined as the smallest value of j such that

$$\frac{\partial}{\partial u(k)} h \circ f^{(j)} \left[x(k), u(k) \right] \neq 0 \tag{4}$$

The nonlinear state feedback that achieves input–output linearization is obtained by solving the following nonlinear algebraic equation for $u(k)$

$$h \circ f^{(r)} \left[x(k), u(k) \right] = v(k) \tag{5}$$

where $v(k)$ is a new input variable

Assumption 1: *The nonlinear algebraic equation* (5) *has a unique solution*

$$u(k) = \Psi \left[x(k), v(k) \right] \quad \Psi(0,0) = 0 \tag{6}$$

for all $x(k) \in R^n$ *and all* $v(k) \in R$.

A necessary condition for Assumption 1 to hold is that the nonlinear system possesses a well defined relative degree throughout the state space. Necessary and sufficient conditions under which the assumption is satisfied are presented elsewhere (Sandberg 1981).

Assumption 2: *There exists a globally defined diffeomorphism* $\left[\xi^{\mathrm{T}}(k), \eta^{\mathrm{T}}(k) \right] = \Phi^{\mathrm{T}} \left[x(k) \right]$, $\Phi(0) = 0$, *such that the nonlinear state feedback* (6) *yields the normal form*

$$\left. \begin{aligned} \xi(k+1) &= A\xi(k) + Bv(k) \\ \eta(k+1) &= q\left[\xi(k), \eta(k), v(k) \right] \\ y(k) &= C\xi(k) \end{aligned} \right\} \tag{7}$$

where the triplet (A, B, C) *is in Brunovsky canonical form and* $q(0,0,0) = 0$.

Sufficient conditions for the existence of a local diffeomorphism are given in Monaco and Normand-Cyrot (1987); we have not found global results for the discrete-time case.

The normal form partitions the system into an r-dimensional linear part and an $(n-r)$-dimensional nonlinear part. We invoke the following assumptions on the nonlinear subsystem.

Assumption 3: *The function* $q\left[\xi(k), \eta(k), v(k) \right]$ *is globally Lipschitz with respect to* $\xi(k), \eta(k)$ *and* $v(k)$.

Assumption 4: *The origin of the zero dynamics* $\eta(k+1) = q\left[0, \eta(k), 0 \right]$ *is globally exponentially stable.*

Sufficient conditions under which Assumption 4 holds are presented in Scokaert and Rawlings (1997). The origin of the linear subsystem can be stabilized with the pole-placement control law

$$v(k) = -\alpha_r \xi_r(k) - \alpha_{r-1} \xi_{r-1}(k) - \cdots - \alpha_1 \xi_1(k) \tag{8}$$

where the α_i are adjustable controller parameters. In the continuous-time case, sufficient conditions under which FLC yields a globally stable closed-loop system are available (Sastry and Isidori 1989). The following theorem provides analogous conditions for discrete-time nonlinear systems.

Theorem 1: *If Assumptions* 1–4 *hold and the parameters* α_i *are chosen such that the roots of the characteristic polynomial* $z^r + \alpha_r z^{r-1} + \cdots + \alpha_1$ *are contained inside the* *unit circle, then* $x(k) = 0$ *is a globally asymptotically stable fixed point of the closed-loop system comprising* (1), (6) *and* (8).

Proof: The normal form (7) is globally defined by Assumptions 1 and 2. The linear state feedback (8) yields a linear subsystem with characteristic polynomial $z^r + \alpha_r z^{r-1} + \cdots + \alpha_1$ (Henson and Seborg 1993). Since this polynomial is stable by construction, it follows that $\lim_{k\to\infty}\xi(k) = 0$ and $\lim_{k\to\infty}v(k) = 0$. The nonlinear subsystem can be rewritten as

$$\begin{aligned}\eta(k+1) &= q\left[0, \eta(k), 0\right] + q\left[\xi(k), \eta(k), v(k)\right] - q\left[0, \eta(k), 0\right] \\ &\equiv q\left[0, \eta(k), 0\right] + p(k)\end{aligned} \tag{9}$$

where $\lim_{k\to\infty}p(k) = 0$ because of Assumption 3 and the convergence of $\xi(k)$ and $v(k)$. By Assumptions 3 and 4, it follows from Lemma 3 in the Appendix that the origin is a globally asymptotically stable fixed point of (9). The inverse transformation $x(k) = \Phi^{-1}\left[\xi(k), \eta(k)\right]$ is globally defined by Assumption 2. Because asymptotic stability is preserved under diffeomorphism, it follows that $\lim_{k\to\infty}x(k) = 0$. □

Assumptions 3 and 4 obviously are not required if the system is fully linearized ($r = n$). It is important to note that Theorem 1 provides sufficient conditions for *asymptotic* stability, while analogous conditions for the continuous-time case ensure only bounded tracking (Sastry and Isidori 1989).

3. Constrained nonlinear systems

We now present an input–output linearization strategy for nonlinear systems subject to input constraints of the form

$$u_{\min} \le u(k) \le u_{\max} \tag{10}$$

where $u_{\min} \le 0 \le u_{\max}$. As in standard FLC, the first step in the controller design involves the application of the nonlinear state feedback (6) and nonlinear change of coordinates such that the normal form (7) is obtained. Instead of employing the pole-placement control law (8), the linear subsystem is stabilized with a linear model predictive controller (LMPC) (Rawlings and Muske 1993) which provides explicit constraint handling. As discussed below, the key step is the transformation of the actual input constraints into input constraints on the feedback linearized system.

The LMPC controller results from the solution of the following open-loop optimal control problem

$$\min_{V(k|k)} \Phi(k) = \sum_{j=0}^{\infty} \xi^{\mathrm{T}}(k+j|k)Q\xi(k+j|k) + rv^2(k+j|k) + s\left[v(k+j|k) - v(k+j-1|k)\right]^2 \tag{11}$$

where $r > 0$ and $s \ge 0$ are scalar tuning parameters, and Q is a positive semidefinite tuning matrix. The decision vector is $V(k|k) = \left[v(k|k) \cdots v(k+N-1|k)\right]^{\mathrm{T}}$, where N is the *control horizon*. Inputs beyond the control horizon are set equal to zero: $v(k+j|k) = 0$ for all $j \ge N$. At each time step, the controller is initialized with the current state measurement: $\xi(k|k) = \left[\Phi_1\left[x(k)\right] \cdots \Phi_r\left[x(k)\right]\right]^{\mathrm{T}}$. A state feedback control law is obtained by implementing only the first calculated input $v(k) = v(k|k)$, and then resolving the problem at the next time step with new state measurements $x(k+1)$. The actual input $u(k)$ is calculated from $v(k)$ via the FLC law (6).

The optimization problem (11) is solved subject to input constraints of the form

$$v_{\min}(k+j|k) \leq v(k+j|k) \leq v_{\max}(k+j|k), \quad 0 \leq j \leq N-1 \tag{12}$$

As discussed below, it may be advantageous to impose the equality constraint

$$\xi(k+N|k) = 0 \tag{13}$$

The resulting problem can be efficiently solved as a quadratic program (Muske and Rawlings 1993b). The input constraints (12) are determined by mapping the original constraints into the feedback linearized space using the nonlinear equation (5)

$$\left.\begin{aligned} v_{\min}(k+j|k) &= \min_u h \circ f^{(r)}\left[x(k+j|k), u\right] \\ v_{\max}(k+j|k) &= \max_u h \circ f^{(r)}\left[x(k+j|k), u\right] \end{aligned}\right\} \tag{14}$$

where $u_{\min} \leq u \leq u_{\max}$ and $x(k+j|k)$ represents an estimate of the future state vector $x(k+j)$. Given $x(k+j|k)$ these optimization problems are easily solved because Assumption 1 implies that $h \circ f^{(r)}\left[x, u\right]$ is a monotonic function of u for all x. However, future state estimates are not available until the LMPC problem is solved, and the LMPC problem cannot be solved until the constraints are specified. As a result, when $N > 1$, exact constraint mapping requires computationally intensive nonlinear programming (Nevistic and Re 1994) or iterative (Oliveira *et al.* 1995) solution methods. As shown below, a much simpler technique is to use inputs calculated at the previous time step to approximate the future constraints. Our computational experience demonstrates this method can yield transformed constraints that are very close to the exact values (Kurtz and Henson 1997b).

Solution of the LMPC problem at the previous time step yields

$$V(k-1|k-1) = \left[v(k-1|k-1) \cdots v(k+N-2|k-1)\right]^{\mathrm{T}} \tag{15}$$

The first input is used to calculate $u(k-1)$, while the remaining inputs are used to construct an estimate of the control sequence at the current time step

$$V(k|k-1) = \left[v(k|k-1) \cdots v(k+N-2|k-1)\ 0\right]^{\mathrm{T}} \tag{16}$$

The current measurement $x(k)$ is used to calculate the transformed state variables $\xi(k)$ and $\eta(k)$ via the nonlinear change of coordinates: $\left[\xi^{\mathrm{T}}(k), \eta^{\mathrm{T}}(k)\right] = \Phi^{\mathrm{T}}\left[x(k)\right]$ Taking these values as initial conditions, the normal form (7) is iterated with the input sequence $V(k|k-1)$ to yield predicted values of the transformed state variables

$$\left.\begin{aligned} \bar{\xi}(k+j|k) &= A\bar{\xi}(k+j-1|k) + Bv(k+j-1|k-1), \\ \bar{\xi}(k|k) &= \xi(k) \\ \bar{\eta}(k+j|k) &= q\left[\bar{\xi}(k+j-1|k), \bar{\eta}(k+j-1|k), v(k+j-1|k-1)\right] \\ \bar{\eta}(k|k) &= \eta(k) \end{aligned}\right\} \tag{17}$$

where $1 \leq j \leq N - 1$ and the overbar denotes state vectors calculated from the estimated input sequence $V(k|k-1)$ rather than the actual sequence $V(k|k)$. These predicted values are used to compute estimated values of the actual state variables via the inverse transformation

$$\begin{bmatrix} \bar{x}(k|k) \\ \bar{x}(k+1|k) \\ \vdots \\ \bar{x}(k+N-1|k) \end{bmatrix} = \begin{bmatrix} x(k) \\ \Phi^{-1}\left[\bar{\xi}(k+1|k), \bar{\eta}(k+1|k)\right] \\ \vdots \\ \Phi^{-1}\left[\bar{\xi}(k+N-1|k), \bar{\eta}(k+N-1|k)\right] \end{bmatrix} \tag{18}$$

The constraints are calculated using the predicted state values $\bar{x}(k+j|k)$ as in (14). The procedure is repeated at the next time step with the input sequence $V(k|k)$ and the measurement $x(k+1)$.

Two possible ways of initializing the constraint mapping algorithm are: (i) extend the first constraint over the entire control horizon (Henson and Kurtz 1994); or (ii) employ a nonlinear programming or iterative solution scheme (Nevistic and Re 1994, Oliveira *et al.* 1995) at the first time step only. The first method is simple, while the second method offers certain theoretical advantages (discussed below). In either case the first set of constraints, $v_{\min}(k|k)$ and $v_{\max}(k|k)$, always map exactly to the actual constraints (10) since they are calculated using the current state measurement. In fact, it is straightforward to show that this property holds even if the feedback linearizing control law and the constraint mapping scheme utilize an estimated value of the current state vector. Therefore, the implemented input $u(k)$ necessarily satisfies the actual constraints (10).

An input sequence $V(k|k)$ is said to be *feasible* if all elements of the sequence remain within the input constraints (12) and the sequence yields a finite value of the objective function $\Phi(k)$. A feasible input sequence must also satisfy the terminal constraint (13) when it is utilized. The LMPC problem (11) is said to be feasible if there exists such a feasible input sequence. For constant input constraints, feasibility of an input sequence at k implies feasibility of the *same* sequence at $k+1$ (Muske and Rawlings 1993a). This result is critical in proving closed-loop stability when LMPC is applied to a constrained linear system (Rawlings and Muske 1993). The feasibility property does not necessarily hold for the proposed method because the input constraints are state dependent. As discussed below, establishing conditions under which this property holds is the key step in stability analysis.

In practice, the feasibility problem can be addressed in the following way (Kurtz and Henson 1997b). If the LMPC problem is infeasible, constraints are dropped on the final input in the control horizon, $v(k+N-1|k)$, and the problem is resolved. If the problem remains infeasible, constraints are dropped on the last two inputs, $v(k+N-2|k)$ and $v(k+N-1|k)$. The process is continued until feasibility is achieved. Also, the constraint mapping strategy is modified so that unconstrained inputs are not used for constraint prediction by extending the last constrained input over the control horizon to obtain the input sequence $V(k|k-1)$. We have found that removing constraints on the final input is usually sufficient to achieve feasibility.

4. Stability analysis

First, closed-loop stability is analysed when the hybrid FLC-MPC strategy is applied to an unconstrained nonlinear system. We show global asymptotic stability is achieved under the same assumptions invoked in Theorem 1 for FLC based on pole placement. The obvious advantage of the proposed method is that input constraints are considered explicitly in the calculation of the nonlinear state feedback control law. Stability analysis for the constrained case is presented below. The following result holds if the terminal constraint (13) is not imposed.

Theorem 2: *If Assumptions 1–4 hold and* $N \geq 1$, *then* $x(k) = 0$ *is a globally asymptotically stable fixed point of the closed-loop system comprising* (1), (6) *and* (11).

Proof: The LMPC problem (11) with $N \geq 1$ is feasible $\forall x(0)$ since the linear subsystem is stable and unconstrained. It follows from Lemma 4 in the Appendix that $\lim_{k\to\infty} \xi(k) = 0$. From the proof of Theorem 1, it follows that $\lim_{k\to\infty} x(k) = 0$. □

A slightly different result is obtained when the terminal constraint (13) is utilized.

Theorem 3: *If Assumptions* 1–4 *hold and* $N \geq r$, *then* $x(k) = 0$ *is a globally asymptotically stable fixed point of the closed-loop system comprising* (1), (6), (11) *and* (13).

Proof: The LMPC problem (11) and (13) is feasible $\forall x(0)$ since the pair (A, B) is controllable, $N \geq r$, and the system is unconstrained. The result then follows from the proofs of Theorems 1 and 2. □

Next we establish additional conditions which ensure the FLC-MPC strategy yields global asymptotic stability when applied to a nonlinear system with input constraints. The next assumption ensures that a feasible input sequence at k can be constructed from the input sequence calculated at $k - 1$.

Assumption 5: *The input sequence* $V(k|k-1)$ *in* (16) *is feasible for all* $k \geq 1$.

The following result demonstrates that the stability problem can be reduced to establishing conditions under which this assumption is satisfied.

Theorem 4: *If Assumptions* 1–5 *hold and the LMPC problem is feasible at* $k = 0$, *then* $x(k) = 0$ *is a globally asymptotically stable fixed point of the closed-loop system comprising* (1), (6), (11) *and* (12).

Proof: The LMPC problem is feasible $\forall k \geq 0$ because it is assumed to be feasible at $k = 0$ and the input sequence $V(k|k-1)$ is feasible $\forall k \geq 1$ by Assumption 5. Let the objective function obtained with $V(k|k-1)$ be denoted $\bar{\Phi}(k)$. The corresponding transformed inputs and transformed state variables are denoted $\bar{v}(k+j|k)$ and $\bar{\xi}(k+j|k)$, respectively. First we show the state variables $\bar{\xi}(k+j|k)$ are identical to the state variables $\xi(k+j|k-1)$ obtained from the solution of the LMPC problem. The state vector $\xi(k|k-1)$ is calculated directly from the input $v(k-1|k-1)$, while the state vector $\bar{\xi}(k|k) = \xi(k)$ is obtained indirectly from the same input via the state feedback control law (6) and the nonlinear change of coordinates $\Phi(x)$. Because the original system (1) and the normal form (7) are related by diffeomorphism (Assumption 2) and non-singular state feedback (Assumption 1), it follows that $\xi(k|k-1) = \bar{\xi}(k|k)$. This implies that $\xi(k+j|k-1) = \bar{\xi}(k+j|k)$ $\forall j \geq 1$ since $v(k+j|k-1) = \bar{v}(k+j|k)$ $\forall j \geq 0$.

Using this result, the objective function $\bar{\Phi}(k)$ can be expressed as follows

$$\bar{\Phi}(k) = \sum_{j=0}^{\infty} \xi^{\mathrm{T}}(k+j|k)Q\xi(k+j|k) + r\bar{v}^2(k+j|k) + s\left[\bar{v}(k+j|k) - \bar{v}(k+j-1|k)\right]^2$$
$$= \sum_{j=0}^{\infty} \xi^{\mathrm{T}}(k+j|k-1)Q\xi(k+j|k-1) + rv^2(k+j|k-1)$$
$$+ s\left[v(k+j|k-1) - v(k+j-1|k-1)\right]^2 \tag{19}$$

The objective function $\Phi(k-1)$ is

$$\Phi(k-1) = \sum_{j=0}^{\infty} \xi^{\mathrm{T}}(k+j-1|k-1)Q\xi(k+j-1|k-1) + rv^2(k+j-1|k-1)$$
$$+ s\left[v(k+j-1|k-1) - v(k+j-2|k-1)\right]^2 \tag{20}$$

Optimization at k yields an objective function $\Phi(k) \le \bar{\Phi}(k)$. Therefore

$$\Phi(k) - \Phi(k-1) \le \bar{\Phi}(k) - \Phi(k-1) = -\xi^{\mathrm{T}}(k-1)Q\xi(k-1)$$
$$- rv^2(k-1) - s\left[v(k-1) - v(k-2)\right]^2 \tag{21}$$

Because $Q \ge 0$, $r > 0$ and $s \ge 0$, it follows that the sequence $\Phi(k)$ is non-increasing. The sequence is also bounded below by zero, therefore it converges. This requires that $\lim_{k\to\infty} v(k) = 0$, which implies $\lim_{k\to\infty} \xi(k) = 0$ since the matrix A in (7) is stable. From the proof of Theorem 1, it follows that $\lim_{k\to\infty} x(k) = 0$ under Assumptions 1–4. □

When a single control move is employed ($N = 1$), Assumption 5 is satisfied if $v(k|k-1) = 0$ is a feasible input. It is interesting to note that the same condition is required to prove stability of nonlinear anti-windup schemes using the circle criterion (Kendi and Doyle 1995). In this case, the condition ensures the existence of a well-defined conic sector which bounds the state-independent input constraints. For the FLC-MPC method, the condition ensures feasibility of a particular input sequence which allows Lyapunov stability results for constrained linear systems to be applied in a straightforward manner.

The remaining task is to determine sufficient conditions under which Assumption 5 holds. This is a difficult problem because the transformed constraints are state dependent and the state variables used to calculate the constraints at $k-1$ and k generally are different because the control horizon N is finite. First we consider the limiting case where $N \to \infty$. An additional assumption concerning the initialization of the constraint mapping algorithm is required.

Assumption 6: *The constraint mapping algorithm is initialized such that the transformed constraints $v_{\min}(j|0)$ and $v_{\max}(j|0)$ are exact for all $j \ge 0$.*

This assumption means the transformed constraints map exactly to original constraints when the actual values of the state vector are utilized. For instance, when $h \circ f^{(r)}[x, u]$ is a monotonically increasing function of u the assumption implies

$$\Psi\left[x(j), v_{\min}(j|0)\right] = u_{\min}, \quad \Psi\left[x(j), v_{\max}(j|0)\right] = u_{\max} \tag{22}$$

for all $j \ge 0$. In theory, Assumption 6 can be satisfied by employing a nonlinear programming or iterative solution technique (Nevistic and Re 1994, Oliveira *et al.*

1995) at the first time step only. The following result provides sufficient conditions for feasibility of the sequence $V(k|k-1)$ when $N \to \infty$.

Lemma 1: *If Assumptions* 1, 2 *and* 6 *hold and* $N \to \infty$, *then the LMPC problem* (11) *yields a feasible input sequence* $V(k|k-1)$ *for all* $k \geq 1$.

The proof is presented in the Appendix. The practical implication of Lemma 1 is that more accurate transformed constraints are obtained as the control horizon is increased and the constraint mapping algorithm is properly initialized. Our computational experience (Kurtz and Henson 1997 a, b) indicates that there usually exists a finite N such that the algorithm converges to the exact constraints and an asymptotically stable closed-loop system is obtained.

As shown in the proof of Lemma 1, special considerations are required to guarantee feasibility of the final input (i.e. 0) in the sequence $V(k|k-1)$. The final input is more likely to cause infeasibilities than the other inputs because it is not an element of the previous sequence $V(k-1|k-1)$. This is supported by our computational experience that infeasibilities can usually be resolved by removing constraints on the final input only. Thus, it is important to explore conditions under which this input is feasible, independent of the other $N-1$ inputs. Furthermore, feasibility of the final input implies asymptotic stability for the $N=1$ case. We restrict this analysis to nonlinear systems which are completely linearizable ($r = n$).

Assumption 7: *There exists an output function* $\tilde{y}(k) = \tilde{h}[x(k)]$, $\tilde{h}(0) = 0$, *and globally defined diffeomorphism* $\xi(k) = \Phi[x(k)]$, $\Phi(0) = 0$, *such that the nonlinear state feedback* (6) *with* $h[x(k)] = \tilde{h}[x(k)]$ *yields a completely linear system*

$$\left.\begin{aligned} \xi(k+1) &= A\xi(k) + Bv(k) \\ \tilde{y}(k) &= C\xi(k) \end{aligned}\right\} \tag{23}$$

where the triplet (A, B, C) *is in Brunovsky canonical form.*

Sufficient conditions for a local solution to this problem are given in Nijmeijer and van der Schaft (1990); we are not aware of similar results for the global case. The following assumption is required to ensure feasibility of the LMPC problem with terminal constraint (13).

Assumption 8: *For all* $\xi(k)$ *there exists a feasible input sequence* $v(k|k), \ldots, v(N^*-1|k)$ *such that* $\xi(k+N^*|k) = 0$.

This assumption implies that the LMPC controller can satisfy the terminal constraint (13) with any control horizon $N \geq N^*$. The following result provides sufficient conditions for feasibility of the final input in the sequence $V(k|k-1)$ for the fully linearizable case.

Lemma 2: *If Assumptions* 1, 7 *and* 8 *hold and* $N \geq N^*$, *then the LMPC problem* (11) *with terminal constraint* (13) *yields a feasible input* $v(k+N-1|k) = 0$ *for all* $k \geq 1$.

The proof is presented in the Appendix. The same conditions cannot be used to prove feasibility for partially linearizable systems due to the presence of the zero dynamics. In this case, driving the transformed state vector $\xi(k)$ to zero does not ensure the actual state vector $x(k)$ goes to zero at the end of the control horizon.

5. Summary and conclusions

A feedback linearizing control strategy for discrete-time nonlinear systems subject to input constraints has been proposed. The actual constraints are transformed into constraints on the input of the feedback linearized system. The constrained linear system is regulated with a linear model predictive controller that provides explicit constraint compensation. Stability analysis is difficult because the transformed constraints are state dependent. The main result demonstrates that the stability problem can be reduced to establishing conditions under which a particular input sequence is feasible. Some sufficient conditions which guarantee feasibility of this sequence have been presented. In addition, a new stability result for unconstrained discrete-time nonlinear systems, which parallels a well know continuous-time result, was derived.

Acknowledgments

Financial support from an NSF Career Development Award (CTS-9501368) and the DuPont Company is gratefully acknowledged.

Appendix

Stability of perturbed nonlinear systems

The following result is proven in Scokaert and Rawlings (1997).

Lemma 3: *Let* $F: R^n \rightarrow R^n$ *satisfy a global Lipschitz condition with* $F(0) = 0$, *and let the origin be a globally exponentially stable fixed point of* $x(k+1) = F\left[x(k)\right]$ *If* $p(k)$ *is an asymptotically convergent sequence, then the origin is an asymptotically stable fixed point of the perturbed system* $x(k+1) = F\left[x(k)\right] + p(k)$.

Stability of linear model predictive control

The following result is proven in Muske and Rawlings (1993a).

Lemma 4: *For stable A and* $N \geq 1, \xi(k) = 0$ *is a globally asymptotically stable solution of the linear model predictive controller with objective function* (11) *and feasible constraints.*

Proof of Lemma 1: Assumptions 1 and 2 are required to ensure the normal form (7) is well defined. For reasons discussed below, the first $N-1$ inputs and the final input in $V(k|k-1)$ are considered separately. The first $N-1$ inputs are feasible if $v_{\min}(k+j|k) \leq v(k+j|k-1) \leq v_{\max}(k+j|k)$ for $0 \leq j \leq N-2$ and $\forall k \geq 1$. The inputs must satisfy the constraints calculated at the previous time step: $v_{\min}(k+j|k-1) \leq v(k+j|k-1) \leq v_{\max}(k+j|k-1)$. Therefore, the inputs are feasible if $v_{\min}(k+j|k-1) = v_{\min}(k+j|k)$ and $v_{\max}(k+j|k-1) = v_{\max}(k+j|k)$. The constraints at $k-1$ and k are calculated from the state vectors $\bar{x}(k+j|k-1)$ and $\bar{x}(k+j|k)$, respectively. Thus, the constraints are equal if $\bar{x}(k+j|k-1) = \bar{x}(k+j|k)$ for $0 \leq j \leq N-2$ and $\forall k \geq 1$.

As shown by (22), a direct implication of Assumption 6 is that $\bar{x}(j|0) = x(j)$ for $0 \leq j \leq N-2$. Therefore, the aforementioned condition will hold by induction if $\bar{x}(k+j|k-1) = x(k+j)$ implies $\bar{x}(k+j|k) = x(k+j)$. This can be shown to be true via a straightforward, but tedious, analysis of the LMPC problem (11) as $N \rightarrow \infty$. This result establishes that the first $N-1$ inputs in $V(k|k-1)$ are feasible for $\forall k \geq 1$. The final task is to show the final input (i.e. 0) in $V(k|k-1)$ is feasible for $\forall k \geq 1$. A necessary condition for the objective function $\Phi(k)$ in (11) to be finite as

$N \to \infty$ is that $v(k+j|k) \to 0$. Thus, zero must be a feasible value for the final input $v(k+N-1|k)$. □

Proof of Lemma 2: Assumptions 1 and 7 are required to ensure the feedback linearized system (23) is well defined. The input is feasible if $v_{\min}(k+N-1|k) \leq 0 \leq v_{\max}(k+N-1|k)\ \forall k \geq 1$. Since $N \geq N^*$, Assumption 8 implies that the sequence $V(k-1|k-1)$ yields $\xi(k+N-1|k-1) = 0$. Because the state values used to calculate the constraints at k are computed from the same input sequence $V(k|k-1)$, it follows that $\bar{x}(k+N-1|k) = \Phi^{-1}\left[\xi(k+N-1|k-1)\right] = 0$. Therefore, the final constraints at k are

$$v_{\min}(k+N-1|k) = \min_u h \circ f^{(r)}[0,u]$$

$$v_{\max}(k+N-1|k) = \max_u h \circ f^{(r)}[0,u]$$

Because $h \circ f^{(r)}[x,u]$ is a monotonic function of u by Assumption 1, $h \circ f^{(r)}[0,0] = 0$, and $u_{\min} \leq 0 \leq u_{\max}$, it follows that $v_{\min}(k+N-1|k) \leq 0 \leq v_{\max}(k+N-1|k)$ $\forall k \geq 1$. □

References

Achhab, M. E., Callier, F. M., and Wertz, V., 1994, Admissible controls and attainable states for a class of nonlinear systems with general constraints. *Int. J. Robust and Nonlinear Control*, **41**, 267–288.

Aguilar, J. L. M., Garcia, R. A., and D'Attellis, C. E., 1996, Exact linearization of nonlinear systems: trajectory tracking with bounded controls and state constraint. *Int. J. Control*, **65**, 455–467.

Alamir, M., and Bornard, G., 1994, On the stability of receding horizon control of nonlinear discrete-time systems. *Systems Control Lett*, **23**, 291–296.

Alvarez, J., Alvarez, J., and Suarez, R., 1991, Nonlinear bounded control for a class of continuous agitated tank reactors. *Chem. Eng. Sci.*, **46**, 3235–3249.

Blanchini, F., and Miani, S., 1996, Constrained stabilization of continuous-time linear systems. *Systems Control Lett*, **28**, 95–102.

Garcia, C. E., 1984, Quadratic dynamic matrix control of nonlinear processes: an application to a batch reactor process. *AIChE Annual Mtg.*, San Francisco.

Grizzle, J. W., and Kokotovic, P. V. (1988). Feedback linearization of sampled-data systems. *IEEE Trans. Autom. Control*, **33**, 857–859.

Henson, M. A., and Kurtz, M. J., 1994, Input–output linearization of constrained nonlinear processes. *AIChE Annual Mtg.*, San Francisco, CA.

Henson, M. A., and Seborg, D. E., 1993, Theoretical analysis of unconstrained nonlinear model predictive control. *Int. J. Control*, **58**, 1053–1080; 1997, Feedback linearizing control. In M. A. Henson and D. E. Seborg (Eds), *Nonlinear Process Control* chapter 4, pp. 149–231 (Englewood Cliffs, NJ; Prentice-Hall).

Isidori, A., 1989, *Nonlinear Control Systems* (New York, NY: Springer-Verlag).

Kendi, T. A., and Doyle, F. J., 1995, An anti-windup scheme for input-output linearization. *Proc. European Control Conf.*, Rome, Italy, pp. 2653–2658; 1997, An anti-windup scheme for input-output linearization. *J. Process Control.* in press.

Kurtz, M. J., and Henson, M. A., 1996, Linear model predictive control of input-output linearized processes with constraints. *Proc. Chemical Process Control V*, Tahoe City, CA; 1997a, Constrained output feedback of a multivariable polymerization reactor. *Proc. American Control Conf.*, Albuquerque, NM; 1997b, Input–output linearizing control of constrained nonlinear processes. *J. Process Control*, **7**, 3–17.

Lee, H.-G., Arapostathis, A., and Marcus, S. I., 1987, Linearization of discrete-time systems. *Int. J. Control*, **45**, 1803–1822.

Mayne, D. Q., and Michalska, H., 1990, Receding horizon control of nonlinear systems. *IEEE Trans. Autom. Control*, **35**, 814–824.

Meadows, E. S., Henson, M. A., Eaton, J. W., and Rawlings, J. B., 1995, Receding horizon control and discontinuous state feedback stabilization. *Int. J. Control*, **62**, 1217–1229.

Monaco, S., and Normand-Cyrot, D., 1987, Minimum-phase nonlinear discrete-time systems and feedback stabilization. *Proc. IEEE Conf. on Decision and Control*, Los Angeles, pp. 979–986.

Muske, K. R., and Rawlings, J. B., 1993a, Linear model predictive control of unstable processes. *J. Process Control*, **3**, 85–96; 1993b, Model predictive control with linear models. *AIChE J.*, **39**, 262–287.

Nevistic, V., and Del Re, L., 1994, Feasible suboptimal model predictive control for linear plants with state dependent constraints. *Proc. American Control Conf.*, pp. 2862–2866.

Nevistic, V., and Morari, M., 1995, Constrained control of feedback-linearizable systems. *Proc. European Control Conf.*, Rome, Italy, pp. 1726–1731.

Nijmeijer, H., and van der Schaft, A. J., 1990, *Nonlinear Dynamical Control Systems* (New York, NY: Springer-Verlag).

Oliveira, S. L., Nevistic, V., and Morari, M., 1995, Control of nonlinear systems subject to input constraints. *Proc. IFAC Symposium on Nonlinear Control Systems Design*, Tahoe City, CA, pp. 15-20.

Pappas, G. J., Lygeros, J., and Godbole, D. N., 1995, Stabilization and tracking of feedback linearizable systems under input constraints. *Proc. IEEE Conf. Decision and Control*, New Orleans, LA, pp. 596–601.

Rawlings, J. B., and Muske, K. R., 1993, The stability of constrained receding horizon control. *IEEE Trans. Autom. Control*, **38**, 1512–1516.

Rawlings, J. B., Meadows, E. S., and Muske, K. R., 1994, Nonlinear model predictive control: a tutorial and survey. *Proc. IFAC Symposium on Advanced Control of Chemical Processes*, Kyoto, Japan, pp. 203–214.

Saberi, A., Lin, Z., and Teel, A. R., 1996, Control of linear systems with saturating actuators. *IEEE Trans. Autom. Control*, **41**, 368–378.

Sandberg, I. W., 1981, Global implicit functions theorems. *IEEE Trans. Circuits Systems*, **28**, 145–149.

Sastry, S., and Isidori, A., 1989, Adaptive control of linearizable systems. *IEEE Trans. Autom. Control*, **34**, 1123–1131.

Scokaert, P. O. M., Rawlings, J. B., and Meadows, E. S., 1997, Discrete-time stability with perturbations: Application to model predictive control. *Automatica*, **33**, 463–470.

Zhu, G.-Y., Kurtz, M. J., and Henson, M. A., 1997, A critical review of constraint compensation techniques for nonlinear process control. *J. Process Control* (submitted).