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Habituating Control for Nonsquare Nonlinear Processes

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A controller design strategy for nonlinear systems with more manipulated inputs than controlled outputs is proposed. The technique is called "habituating control" as a result of its similarity to control schemes commonly used in biological systems. Nonlinear control laws that provide input−output linearization while simultaneously minimizing the "cost" of affecting control are derived. In the single-output case, the cost function employed differs according to the relative degrees of the two inputs. Local stability analysis shows that the resulting controller can provide a simple solution to the singularity and nonminimum phase problems. An extension of the controller design strategy for multiple-output processes also is presented. The proposed method is evaluated using nonlinear models of chemical and biochemical reaction systems.

1. Introduction

Feedback control systems typically employ equal numbers of manipulated inputs and controlled outputs. In many applications, superior performance and robustness can be achieved by introducing additional input or output variables. A well-established example of this approach is cascade control, where a second output measurement allows improved disturbance rejection using the existing manipulated input. Also widely studied is the introduction of additional input variables to form a nonsquare system with more inputs than outputs. A variety of linear controller design techniques have been proposed for both the single-output (Henson *et al.*, 1995; Popiel *et al.*, 1986; Shinskey, 1978) and multiple-output (Medanic, 1993; Muske and Rawlings, 1993; Siljak, 1980) cases.

Significantly fewer results are available for nonlinear systems with more inputs than outputs. The design of input−output linearizing controllers for nonminimum phase nonlinear systems with a single output and two inputs has been investigated (Doyle *et al.*, 1992; McClamroch and Schumacher, 1993). The first input is used to achieve input−output linearization, while the second input is used to stabilize the otherwise unstable zero dynamics. Nonlinear model predictive control (Bequette, 1991; Rawlings *et al.*, 1994) provides a systematic means for handling nonsquare nonlinear systems with multiple outputs. However, this method has several disadvantages including large computational requirements.

In this paper, we propose an input−output linearizing control strategy for nonsquare nonlinear processes with more manipulated inputs than controlled outputs. The underlying premise is that the control objectives can be satisfied more easily by utilizing additional input variables. Because the additional inputs provide extra degrees of freedom, the nonlinear controller is designed to provide input−output linearization at the *minimum cost*. As explained below, the technique is called "habituating control" due to its similarity to control schemes utilized in biological systems.

The remainder of the paper is organized as follows. The biological motivation for the habituating control strategy is discussed in section 2. Section 3 contains a detailed presentation of the nonlinear controller design technique for the single-output case. An extension for multiple-output processes is presented in section 4. The proposed method is evaluated via two simulation examples in section 5. Finally, section 6 contains a summary and conclusions.

2. Biological Motivation

The baroreceptor reflex (baroreflex) is responsible for short-term regulation of arterial blood pressure (Kirchheim, 1976; Sagawa, 1983). Blood pressure measurements are provided by baroreceptor neurons located within the arterial walls. The pressure measurements are integrated with other cardiorespiratory signals in the lower brain stem. The integrated sensory information is processed by two distinct neural controllers, the parasympathetic and sympathetic nervous systems. These controllers maintain blood pressure at the desired value by manipulating cardiac output and vascular resistance. The parasympathetic system affects only cardiac output, while the sympathetic system primarily affects vascular resistance. As a result, the effect of the parasympathetic system on arterial pressure is quite rapid as compared to that of the sympathetic system. However, sustained variations in cardiac output are physiologically "expensive" as compared to long-term variations in peripheral resistance. Therefore, the parasympathetic system is used preferentially during transient conditions, while the sympathetic system is primarily responsible for steady-state control.

In conjunction with collaborators from DuPont, the third author has developed *linear habituating control strategies* by "reverse engineering" this biological control system (Henson *et al.*, 1995). The technique is applicable to linear systems with a single controlled output and two manipulated inputs which differ in terms of their relative costs and dynamic effects. The primary input, which is analogous to the vascular resistance, is used mainly for steady-state control. The secondary input is analogous to the cardiac output; it is used only during transients under normal operating conditions. Controller design techniques based on direct synthesis and model predictive control have been developed. Simulation results demonstrate the superior performance that can be achieved with habituating control

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as compared to conventional techniques that utilize a single manipulated input. In this paper, we propose a habituating control strategy for processes described by nonlinear process models.

3. Single-Output Processes

Initially we consider the following class of nonlinear systems,

$$\dot{x} = f(x) + g_1(x)\, u_1 + g_2(x)\, u_2 \qquad (1)$$

$$y = h(x)$$

where x is an n-dimensional vector of state variables, u_1 and u_2 are scalar manipulated input variables, and y is a scalar output variable. We assume that the state vector is measured or estimated from available measurements. The objective is to design nonlinear feedback control laws for u_1 and u_2 such that input–output linearization is achieved. The following notation (Henson and Seborg, 1991; Isidori, 1989; Kravaris and Kantor, 1990) is useful. The Lie derivative of the scalar field $h(x)$ with respect to the vector field $f(x)$ is defined as:

$$L_f h(x) \equiv \left[\frac{\partial h(x)}{\partial x}\right]^{\mathrm{T}} f(x) \qquad (2)$$

Higher-order Lie derivatives are defined recursively:

$$L_f^k h(x) \equiv \left[\frac{\partial L_f^{k-1} h(x)}{\partial x}\right]^{\mathrm{T}} f(x) \qquad (3)$$

The ith input u_i has relative degree r_i at the point x_0 if r_i is the smallest integer such that $L_{g_i} L_f^{r_i-1} h(x_0) \neq 0$.

Utilizing both inputs to achieve input–output linearization offers several potential advantages as compared to the standard approach of using a single input. As discussed below, singular points (Henson and Seborg, 1992; Hirschorn and Davis, 1987) and unstable zero dynamics (Doyle *et al.*, 1996; Hauser *et al.*, 1992a) may preclude exact linearization using the primary input u_1. We show that it may be possible to overcome such obstructions by introducing a secondary input u_2. It is important to note that the linearization objective does not yield unique control laws; an additional objective must be specified to obtain a well-defined control problem. We design the control laws to minimize a performance index which corresponds to the *cost* of affecting control. The index differs according to the relative degrees of the two inputs.

3.1. Equal Relative Degrees. First we assume that the two inputs have equal relative degrees: $r_1 = r_2 \equiv r$. In this case, the rth derivative of the output can be represented as:

$$y^{(r)} = L_f^r h(x) + L_{g_1} L_f^{r-1} h(x)\, u_1 + L_{g_2} L_f^{r-1} h(x)\, u_2 \qquad (4)$$

Without loss of generality, we use the first input u_1 to achieve input–output linearization:

$$u_1 = \frac{v - L_f^r h(x) - L_{g_2} L_f^{r-1} h(x)\, u_2}{L_{g_1} L_f^{r-1} h(x)} \qquad (5)$$

Under this control law, the closed-loop system has a linear input–output map: $y^{(r)} = v$. Consequently, the new input v can be designed to place the closed-loop poles and to provide integral action for offset-free tracking,

$$v = -\alpha_r L_f^{r-1} h(x) - \ldots - \alpha_2 L_f h(x) + \alpha_1 [y_{\mathrm{sp}} - h(x)] + \alpha_0 \int_0^t [y_{\mathrm{sp}} - h(x)]\, \mathrm{d}\tau \qquad (6)$$

where y_{sp} is the setpoint and α_i are controller tuning parameters chosen such that the characteristic polynomial $s^{r+1} + \alpha_r s^r + \ldots + \alpha_1 s + \alpha_0$ is Hurwitz.

Note that the input–output linearizing control law (5) is a function of the second input u_2. The objective is to design the control law for u_2 such that the *cost* associated with affecting control is minimized. It is important to note that this approach is more general, and more biologically plausible (Schmidt *et al.*, 1971), than linear controller design techniques (Henson *et al.*, 1995; Popiel *et al.*, 1986; Shinskey, 1978) in which the more "expensive" input is returned to its resting value at steady state. We propose the following cost function:

$$I = \frac{1}{2}\gamma_1^2 (u_1 - \bar{u}_1)^2 + \frac{1}{2}\gamma_2^2 (u_2 - \bar{u}_2)^2 \qquad (7)$$

where $\bar{u}_i$ and γ_i are the desired steady-state value and the "cost" of input u_i, respectively. Note that the cost function penalizes *instantaneous* deviations of the inputs from their steady-state values. Minimizing I with respect to u_2 yields

$$\frac{\mathrm{d}I}{\mathrm{d}u_2} = 0 = \gamma_1^2 (u_1 - \bar{u}_1) \frac{\partial u_1}{\partial u_2} + \gamma_2^2 (u_2 - \bar{u}_2) \qquad (8)$$

where

$$\frac{\partial u_1}{\partial u_2} = -\frac{L_{g_2} L_f^{r-1} h(x)}{L_{g_1} L_f^{r-1} h(x)} \qquad (9)$$

Solving (5) and (8) for u_2 yields the following state feedback control law:

$$u_2 = \frac{\alpha L_{g_2} L_f^{r-1} h(x)}{[L_{g_1} L_f^{r-1} h(x)]^2 + \alpha [L_{g_2} L_f^{r-1} h(x)]^2} [v - L_f^r h(x)] + \frac{[L_{g_1} L_f^{r-1} h(x)]^2 \bar{u}_2 - \alpha L_{g_1} L_f^{r-1} h(x) L_{g_2} L_f^{r-1} h(x)\, \bar{u}_1}{[L_{g_1} L_f^{r-1} h(x)]^2 + \alpha [L_{g_2} L_f^{r-1} h(x)]^2} \qquad (10)$$

where $\alpha = \gamma_1^2/\gamma_2^2$. In practice, α may be employed as a tuning parameter that determines the relative contributions of the two inputs. By substituting (10) into (5), the following control law for u_1 is obtained:

$$u_1 = \frac{L_{g_1} L_f^{r-1} h(x)}{[L_{g_1} L_f^{r-1} h(x)]^2 + \alpha [L_{g_2} L_f^{r-1} h(x)]^2} [v - L_f^r h(x)] + \frac{\alpha [L_{g_2} L_f^{r-1} h(x)]^2 \bar{u}_1 - L_{g_1} L_f^{r-1} h(x)\, L_{g_2} L_f^{r-1} h(x)\, \bar{u}_2}{[L_{g_1} L_f^{r-1} h(x)]^2 + \alpha [L_{g_2} L_f^{r-1} h(x)]^2} \qquad (11)$$

It is insightful to examine the control laws (10) and (11) for limiting values of the tuning parameter α. In the limit as $\alpha \to 0$, $u_2 = \bar{u}_2$ and the control law for u_1 becomes:

$$u_1 = \frac{1}{L_{g_1}L_f^{r-1}h(x)}[v - L_f^r h(x)] - \frac{L_{g_2}L_f^{r-1}h(x)}{L_{g_1}L_f^{r-1}h(x)}\bar{u}_2 \quad (12)$$

This corresponds to the case where the cost associated with manipulating u_2 is much higher than the cost of manipulating u_1. Consequently, u_1 provides the linearizing feedback, while u_2 is maintained at its steady-state value. In the limit as $\alpha \to \infty$, $u_1 = \bar{u}_1$ and the control law for u_2 becomes:

$$u_2 = \frac{1}{L_{g_2}L_f^{r-1}h(x)}[v - L_f^r h(x)] - \frac{L_{g_1}L_f^{r-1}h(x)}{L_{g_2}L_f^{r-1}h(x)}\bar{u}_1 \quad (13)$$

In this case, the cost of manipulating u_1 is much higher than the cost of manipulating u_2. Therefore, u_2 is active and u_1 is held at its steady-state value.

The proposed control strategy can provide a simple and effective means for overcoming the singularity problem. The point x_0 is termed a *singular point* with respect to u_i if $L_{g_i}L_f^{r-1}h(x_0) = 0$, but $L_{g_i}L_f^{r-1}h(x) \neq 0$ for some points x in a neighborhood of x_0 (Henson and Seborg, 1992; Hirschorn and Davis, 1987). Standard input−output linearizing control laws based on a single input u_i are not well-defined on the singularity manifold, which is defined as: $M_s = \{x\colon L_{g_i}L_f^{r-1}h(x) = 0\}$. As an illustration, consider the linearizing control law (12) obtained when u_2 is held at its steady-state value. Because the term $L_{g_1}L_f^{r-1}h(x)$ appears in the denominator, the controller produces unbounded values of u_1 on the singularity manifold. Although design techniques for systems with singularities have been proposed, the resulting control laws provide only approximate linearization and/or are difficult to analyze (Castillo, 1991; Crouch *et al.*, 1991; Hauser *et al.*, 1992b). By contrast, singularities are handled easily with the habituating control strategy. On the singularity manifold where $L_{g_1}L_f^{r-1}h(x) = 0$, $u_1 = \bar{u}_1$ and the control law (13) with the last term vanishing is obtained for u_2. At points where $L_{g_2}L_f^{r-1}h(x) = 0$, $u_2 = \bar{u}_2$ and the control law for u_1 is (12) with the last term vanishing. Note that the control laws are not well-defined at points where $L_{g_1}L_f^{r-1}h(x) = L_{g_2}L_f^{r-1}h(x) = 0$.

3.2. Different Relative Degrees. Now the controller design procedure is generalized to systems in which the two inputs have different relative degrees. Without loss of generality, assume that the relative degree of the first input is less than the relative degree of the second input: $r_1 < r_2$. When computing derivatives of the output, we assume that u_2 appears via the vector field $f(x)$ rather than $g_1(x)$. This simplifies the controller design since it ensures that u_2 will first appear in the r_2th derivative via the function $L_{g_2}L_f^{r_2-1}h(x)$.

The first step is to construct an *extended system* (Henson and Seborg, 1990; Nijmeijer and van der Schaft, 1990) in which the two manipulated inputs have the same relative degree. The extended system is obtained by introducing $\mu = r_2 - r_1$ integrators into the u_1 input channel,

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{\mu-1} &= z_\mu \\ \dot{z}_\mu &= w_1 \\ u_1 &= z_1 \end{aligned} \quad (14)$$

where z_i represent controller state variables and w_1 is a new manipulated input that replaces u_1 in the controller design. The extended system has the following state-space representation:

$$\begin{bmatrix} \dot{x} \\ \dot{z}_1 \\ \vdots \\ \dot{z}_{\mu-1} \\ \dot{z}_\mu \end{bmatrix} = \begin{bmatrix} f(x) + g_1(x)\, z_1 \\ z_2 \\ \vdots \\ z_\mu \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} w_1 + \begin{bmatrix} g_2(x) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u_2 \quad (15)$$

$$y = h(x)$$

By defining $x_e = [x^T\ z^T]^T$ and $w_2 = u_2$, the extended system can be rewritten as:

$$\dot{x}_e = f_e(x_e) + g_{1e}(x_e)\, w_1 + g_{2e}(x_e)\, w_2 \quad (16)$$

$$y = h_e(x_e)$$

By construction, the manipulated inputs w_1 and w_2 have the same relative degree $r = r_2$. The extended system can be used to design the nonlinear control laws as described in the previous section, where the form of the input v in (6) is modified accordingly. It is important to remember that $w_1 = u_1^{(\mu)}$ when analyzing the resulting control laws. Based on the equal relative degree case, the following results are easily derived:

1. In the limit as $\alpha \to 0$, $u_2 = \bar{u}_2$ and u_1 provides the linearizing feedback.
2. In the limit as $\alpha \to \infty$, $u_1^{(\mu)} = \bar{u}_1^{(\mu)} = 0$. If $u_1(0) = \bar{u}_1$ and the system is initially at rest, then $u_1 = \bar{u}_1$ and u_2 provides the linearizing feedback.
3. At points where $L_{g_1}L_f^{r_1-1}h(x) = 0$, $u_1^{(\mu)} = \bar{u}_1^{(\mu)} = 0$ and *both* inputs contribute to the linearizing feedback. If $\mu = 1$, then u_1 is held *constant* on the singularity manifold.
4. At points where $L_{g_2}L_f^{r_2-1}h(x) = 0$, $u_2 = \bar{u}_2$ and u_1 provides the linearizing feedback.
5. At steady state, $u_2 = \bar{u}_2$ and u_1 maintains the output at the setpoint.

By analogy to linear habituating control (Henson *et al.*, 1995), the final result shows that u_1 and u_2 can be identified as the primary input and secondary input, respectively.

3.3. Local Stability. Next we perform a local stability analysis of the closed-loop system obtained with the habituating controller. Of particular interest is the case where the zero dynamics associated with one of the inputs is unstable. Standard input−output linearization techniques based on a single input cannot be applied to such *nonminimum phase* systems. Below we show that nonlinear habituating control can provide an effective method for overcoming the nonminimum phase problem.

The zero dynamics associated with the input u_1 are constructed as follows. Under the linearizing control law (12), there exists a nonlinear coordinate transformation $[\xi^T, \eta^T] = \Phi^T(x)$ such that the nonlinear system (1) has the following normal form representation (Isidori, 1989):

$$\dot{\xi} = A\xi + Bv$$
$$\dot{\eta} = q(\xi,\eta) \tag{17}$$
$$y = C\xi$$

The zero dynamics are defined as dynamics of the $(n - r_1)$-dimensional nonlinear subsystem when the variables of the r_1-dimensional linear subsystem $\xi = 0$:

$$\dot{\eta} = q(0,\eta) \tag{18}$$

Stability of the zero dynamics is a necessary and sufficient condition for the control law (12) to yield local closed-loop stability (Byrnes and Isidori, 1988). The zero dynamics associated with the input u_2 can be derived similarly. Note that the introduction of integrators does not affect the stability of the zero dynamics (Isidori, 1989).

Stability analysis is based on the Jacobian approximation of the extended nonlinear system (16):

$$\dot{x}_e = Ax_e + b_1w_1 + b_2w_2 \tag{19}$$
$$y = cx_e$$

The input–output behavior of the linearized system can be represented as,

$$y(s) = \frac{\beta_{n-r_1}s^{n-r_1} + \beta_{n-r_1-1}s^{n-r_1-1} + ... + \beta_1 s + \beta_0}{s^{\mu}(s^n + \alpha_{n-1}s^{n-1} + ... + \alpha_1 s + \alpha_0)}w_1(s) + \frac{\hat{\beta}_{n-r_2}s^{n-r_2} + \hat{\beta}_{n-r_2-1}s^{n-r_2-1} + ... + \hat{\beta}_1 s + \hat{\beta}_0}{s^n + \alpha_{n-1}s^{n-1} + ... + \alpha_1 s + \alpha_0}w_2(s)$$
$$\equiv \frac{N(s)}{s^{\mu}D(s)}w_1(s) + \frac{\hat{N}(s)}{D(s)}w_2(s) \tag{20}$$

where $N(s)/D(s)$ and $\hat{N}(s)/D(s)$ are (possibly nonminimal) realizations of the transfer functions $y(s)/u_1(s)$ and $y(s)/u_2(s)$, respectively. It is important to note that the zeros of the transfer function $y(s)/u_1(s)$ are identically equal to the eigenvalues of the linearized version of the zero dynamics (18) (Isidori, 1989). Similarly, the zeros of the transfer function $y(s)/u_2(s)$ equal the eigenvalues of the linearized zero dynamics associated with u_2.

Theorem 1. If the characteristic polynomial

$$\beta_{n-r_1}N(s) + \alpha\hat{\beta}_{n-r_2}s^{\mu}\hat{N}(s) = 0 \tag{21}$$

associated with the linearized zero dynamics of the extended nonlinear system (16) is Hurwitz, then the habituating controller is locally stabilizing.

The proof is presented in the Appendix.

If the two inputs have equal relative degrees ($\mu = 0$), Theorem 1 shows that the linearized zero dynamics of the single-input, single-output systems u_1/y and u_2/y are recovered in the limit as $\alpha \to 0$ and $\alpha \to \infty$, respectively. By contrast, only the zero dynamics of u_1/y can be recovered when the relative degrees are different ($\mu \geq 1$). Corollary 1 follows directly from these observations.

Corollary 1. There exists a tuning parameter $\alpha \in [0, \infty)$ *such that the habituating controller is locally stabilizing if (i) the two inputs have equal relative degrees and the linearized zero dynamics associated with either u_1 or u_2 is stable or (ii) the two inputs have different relative degrees and the linearized zero dynamics associated with u_1 is stable.*

For systems that do not satisfy the conditions of Corollary 1, there may exist values of α such that (21) is a Hurwitz polynomial and the closed-loop system is locally stable. Corollary 2 provides a necessary condition for the existence of a stabilizing α.

Corollary 2. *If the two inputs have different relative degrees and the linearized zero dynamics associated with u_1 are unstable, then the habituating controller is locally stabilizing only if: (i) the linearized zero dynamics associated with u_1 and u_2 do not have common eigenvalues with positive real part; and (ii) the linearized zero dynamics associated with u_1 has an even number of eigenvalues with positive real part.*

The first condition follows directly from (21), while a proof for the second condition is presented in the Appendix.

4. Multiple-Output Processes

Now we generalize the nonlinear habituating control technique to multiple-output systems of the form

$$\dot{x} = f(x) + g(x)\,u \tag{22}$$
$$y = h(x)$$

where x is an n-dimensional vector of state variables, u is an m-dimensional vector of manipulated inputs, and y is a p-dimensional vector of controlled outputs. We assume that the number of inputs is strictly greater than the number of outputs ($m > p$). The objective is to design state feedback control laws such that the input–output response is both linear and decoupled and the cost of affecting control is minimized.

The ith output is said to have relative degree r_i at the point x_0 if r_i is the smallest integer such that $L_{g_j}L_f^{r_i-1}h_i(x_0) \neq 0$ for at least one $j \in [1, m]$. Therefore, time derivatives of the outputs can be represented as (Isidori, 1989):

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_p^{(r_p)} \end{bmatrix} = \begin{bmatrix} L_f^{r_1}h_1(x) \\ \vdots \\ L_f^{r_p}h_p(x) \end{bmatrix} + \begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1(x) & \cdots & L_{g_m}L_f^{r_1-1}h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1}L_f^{r_p-1}h_p(x) & \cdots & L_{g_m}L_f^{r_p-1}h_p(x) \end{bmatrix} u \equiv b(x) + A(x)\,u \tag{23}$$

We assume that the rank of the matrix $A(x)$ at the point x_0 is greater than or equal to p. This is a necessary and sufficient condition for achieving local input–output decoupling with static state feedback (Isidori, 1989). Under this assumption, the input vector can be partitioned such that

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_p^{(r_p)} \end{bmatrix} = b(x) + A_1(x)\, u_1 + A_2(x)\, u_2 \tag{24}$$

where u_1 is a p-dimensional vector, u_2 is an $(m - p)$-dimensional vector, $A_1(x)$ is a $p \times p$ matrix that is invertible at x_0, and $A_2(x)$ is a $p \times m - p$ matrix. Note that the partitioning may not be unique. The input–output decoupling control law is obtained by setting the output derivatives equal to an m-dimensional vector of new inputs and solving the resulting equation for u_1:

$$u_1 = A_1^{-1}(x)\,[v - b(x) - A_2(x)\, u_2] \tag{25}$$

Under this control law, the closed-loop system has a linear and decoupled input–output map: $y_i^{(r_i)} = v_i$. Consequently, the new input v_i can be designed as in the single-output case.

We use the additional manipulated inputs u_2 to design a second state feedback control law that minimizes the cost of affecting control. The cost function utilized is

$$I = \frac{1}{2}(u_1 - \bar{u}_1)^{\mathrm{T}}\boldsymbol{\Gamma}_1(u_1 - \bar{u}_1) + \frac{1}{2}(u_2 - \bar{u}_2)^{\mathrm{T}}\boldsymbol{\Gamma}_2(u_2 - \bar{u}_2) \equiv I_1 + I_2 \tag{26}$$

where $\bar{u}_1$ is the desired steady-state value of u_1 and $\boldsymbol{\Gamma}_1$ is a diagonal matrix with non-negative elements that correspond to the cost of manipulating the individual inputs. The vector $\bar{u}_2$ and matrix $\boldsymbol{\Gamma}_2$ represent analogous quantities for u_2. Minimizing I with respect to u_2 yields

$$\frac{\mathrm{d}I}{\mathrm{d}u_2} = \frac{\partial I_1}{\partial u_1}\frac{\partial u_1}{\partial u_2} + \frac{\partial I_2}{\partial u_2} = -(A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1(u_1 - \bar{u}_1) + \boldsymbol{\Gamma}_2(u_2 - \bar{u}_2) = 0 \tag{27}$$

where the state dependence has been omitted for simplicity. The state feedback control laws result from simultaneous solution of the two sets of equations (25) and (27):

$$\begin{bmatrix} A_1 & A_2 \\ -(A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v - b \\ -(A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1\bar{u}_1 + \boldsymbol{\Gamma}_2\bar{u}_2 \end{bmatrix} \tag{28}$$

The habituating controller exists if and only if these equations have a unique solution at x_0. A proof of the following theorem is presented in the Appendix:

Theorem 2. *The habituating controller exists if rank*$[A_1(x_0)] = p$ *and either of the following conditions hold: (i) rank*$(\boldsymbol{\Gamma}_2) = m - p$ *or (ii) rank*$(\boldsymbol{\Gamma}_1) = p$ *and rank*$[A_1^{-1}(x_0)\, A_2(x_0)] = m - p$.

Under these conditions, (28) can be solved to yield

$$u_1 = [A_1^{-1} - (A_1^{-1}A_2)\boldsymbol{\Delta}^{-1}(A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1A_1^{-1}](v - b) + (A_1^{-1}A_2)\boldsymbol{\Delta}^{-1}(A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1\bar{u}_1 - (A_1^{-1}A_2)\boldsymbol{\Delta}^{-1}\boldsymbol{\Gamma}_2\bar{u}_2 \tag{29}$$

$$u_2 = \boldsymbol{\Delta}^{-1}(A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1A_1^{-1}(v - b) - \boldsymbol{\Delta}^{-1}(A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1\bar{u}_1 + \boldsymbol{\Delta}^{-1}\boldsymbol{\Gamma}_2u_2$$

where the matrix $\boldsymbol{\Delta} \equiv \boldsymbol{\Gamma}_2 + (A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1(A_1^{-1}A_2)$ is invertible at x_0. It is interesting to note that the nonlinear habituating controller reduces to the standard input–output decoupling controller (Isidori, 1989) when the cost associated with manipulating u_2 is much higher than the cost of manipulating u_1 ($\boldsymbol{\Gamma}_1 = 0$). In this case, the control law (29) yields (25) with $u_2 = \bar{u}_2$. A more complicated result is obtained if the cost of manipulating u_1 is very high as compared to the cost of manipulating u_2 ($\boldsymbol{\Gamma}_2 = 0$). Both u_1 and u_2 are needed to achieve input–output decoupling, in general, although only u_2 is utilized if there are twice as many inputs as outputs ($m = 2p$). Note that this condition is satisfied in the single-output case.

The habituating control technique can be advantageous for processes that have singular decoupling matrices with respect to the primary inputs (Kravaris and Soroush, 1990). In this case, secondary inputs are introduced and the input vector is partitioned such that the system is input–output decouplable. Then the weighting matrices ($\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_2$) are used to determine the relative contribution of the individual inputs. This represents a very simple and effective approach as compared to standard input–output decoupling techniques, which do not utilize all the available inputs (Isidori, 1989) or produce a complicated dynamic control law using just the primary inputs (Nijmeijer and Respondek, 1988). We currently are investigating the stability properties of the habituating controller in the multiple-output case.

5. Simulation Examples

Chemical Reactor. First we apply the habituating control strategy to a nonlinear chemical reactor. The process model describes a reversible reaction $A \rightleftharpoons B$ that occurs in a constant-volume, stirred-tank reactor (Economou *et al.*, 1986),

$$\dot{C}_A = \frac{q}{V}(C_{Ai} - C_A) - k_1(T)\, C_A + k_2(T)\, C_B$$

$$\dot{C}_B = \frac{q}{V}(C_{Bi} - C_B) + k_1(T)\, C_A - k_2(T)\, C_B \tag{30}$$

$$\dot{T} = \frac{q}{V}(T_i - T) + \frac{(-\Delta H)}{\rho C_p}[k_1(T)\, C_A - k_2(T)\, C_B]$$

where $k_1(T) = C_1 \exp(-E_1/RT)$ and $k_2(T) = C_2 \exp(-E_2/RT)$. Symbol definitions and nominal operating conditions are given in Table 1. Economou *et al.* (1986) have designed a nonlinear internal model controller for this system using the feed temperature T_i and effluent concentration C_B as the manipulated input (u_1) and controlled output (y), respectively. For this choice of variables, the relative degree $r_1 = 2$ and the linearizing control law is singular on the manifold:

$$C_B = \frac{E_1k_1(T)}{E_1k_1(T) + E_2k_2(T)} \tag{31}$$

Table 1. Nominal Operating Conditions for Chemical Reactor

symbol	definition	nominal value
q	inlet flow rate	1 L/s
C_{Ai}	inlet concentration of A	1 mol/L
C_{Bi}	inlet concentration of B	0 mol/L
T_i	inlet temperature	392.4 K
V	reactor volume	60 L
C_1	preexponential factor for forward reaction	5×10^3 s^{-1}
C_2	preexponential factor for reverse reaction	1×10^6 s^{-1}
E_1	activation energy for forward reaction	1×10^4 cal/mol
E_2	activation energy for reverse reaction	1.5×10^4 cal/mol
$-\Delta H$	heat of reaction	5000 cal/mol
ρ	density	1 kg/L
C_p	heat capacity	1000 cal/kg·K
C_A	effluent concentration of A	0.6 mol/L
C_B	effluent concentration of B	0.4 mol/L
T	reactor temperature	394.4 K

It is interesting to note that the optimum conversion ($C_B/C_{Ai} = 0.508$) belongs to the singularity manifold. Consequently, input−output linearization should not be applied if T_i is the only input variable.

We consider introducing an additional manipulated input to overcome the singularity problem. Assume that operational requirements dictate that the throughput q remain constant. In this case, the inlet concentration C_{Ai} may be chosen as the second input. However, large excursions of C_{Ai} from its nominal value may lead to increased raw material costs. Therefore, it is desirable to use both T_i and C_{Ai} as manipulated inputs. It is easy to show that C_{Ai} has relative degree $r_2 = 2$ and the linearized zero dynamics associated with both inputs is stable for all operating points of interest.

We compare a nonlinear habituating controller (NHC) that manipulates both T_i and C_{Ai} and an input−output linearizing controller (IOLC) that manipulates only T_i. In both cases, the input v is designed to yield the closed-loop characteristic polynomial $(\epsilon s + 1)^3 = 0$, where $\epsilon = 15$ s. NHC is tuned with $\alpha = 5 \times 10^{-5}$, $\bar{T}_i = 392.4$ K, and $\bar{C}_{Ai} = 1$ mol/L. The controllers are compared for a setpoint change to the optimum conversion (where $C_B = 0.508$ g/L) in Figure 1. The controllers appear to yield the same output response. However, IOLC produces very large control moves as the singularity at the optimum is approached. As a result, the simulation fails completely at approximately $t = 185$ s. NHC generates reasonable T_i changes by employing C_{Ai} as an additional manipulated input. Note that only small variations in C_{Ai} are needed to avoid the singularity. Figure 1 also shows that NHC can handle much larger setpoint changes ($C_B = 0.6$ g/L) without requiring large control moves.

In Figure 2, the controllers are compared for a sudden change in the activation energy of the second reaction ($E_2 = 1.44 \times 10^4$ cal/mol) while operating at a constant setpoint. IOLC is unable to handle the disturbance because the singularity manifold is encountered. As a result, the simulation fails at $t = 310$ s. NHC provides smooth disturbance rejection and reasonable control moves. Note that a faster response could be obtained by retuning the controller.

In Figures 3 and 4, we utilize the following initial conditions: $C_A(0) = C_B(0) = 0.5$ g/L, $T(0) = 424.9$, $T_i(0) = 422.4$ K. Note that these values correspond to a steady state much closer to the optimum conversion than the steady state in Table 1. NHC is retuned with

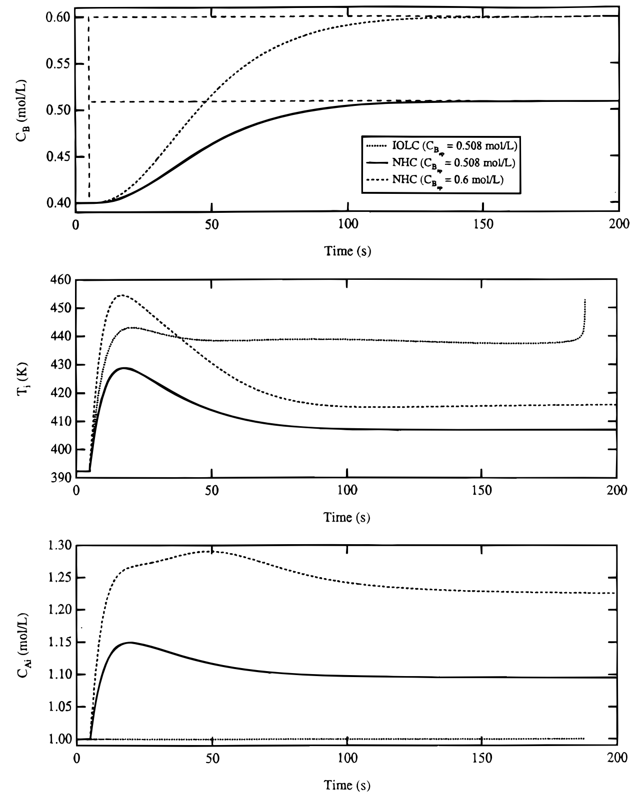

Figure 1. IOLC and NHC for setpoint changes (chemical reactor).

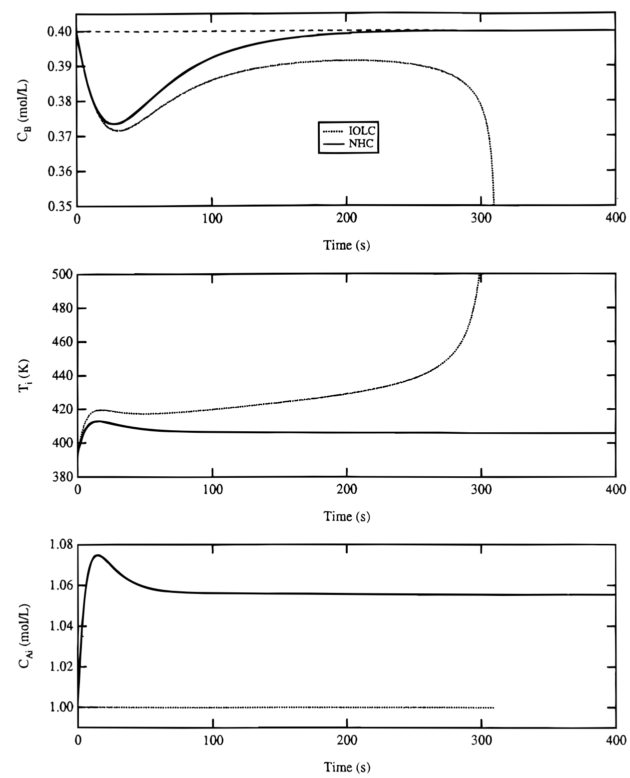

Figure 2. IOLC and NHC for a sudden change in the activation energy E_2 (chemical reactor).

$\bar{T}_i = 422.4$ K to account for the initial condition change. Figure 3 compares NHC and IOLC for a step disturbance in the inlet flow rate ($q = 1.10$ L/s). IOLC is unable to handle the disturbance because a singularity is encountered, and the simulation fails at $t = 100$ s. By contrast, NHC provides effective disturbance rejection by employing C_{Ai} as an additional manipulated input. Figure 3 also shows that NHC can handle much

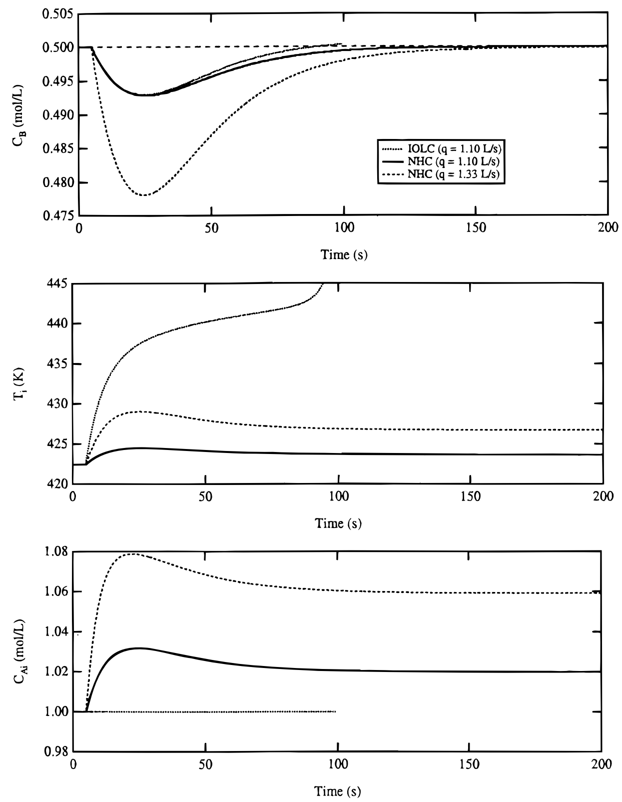

Figure 3. IOLC and NHC for step disturbances in the inlet flow rate (chemical reactor).

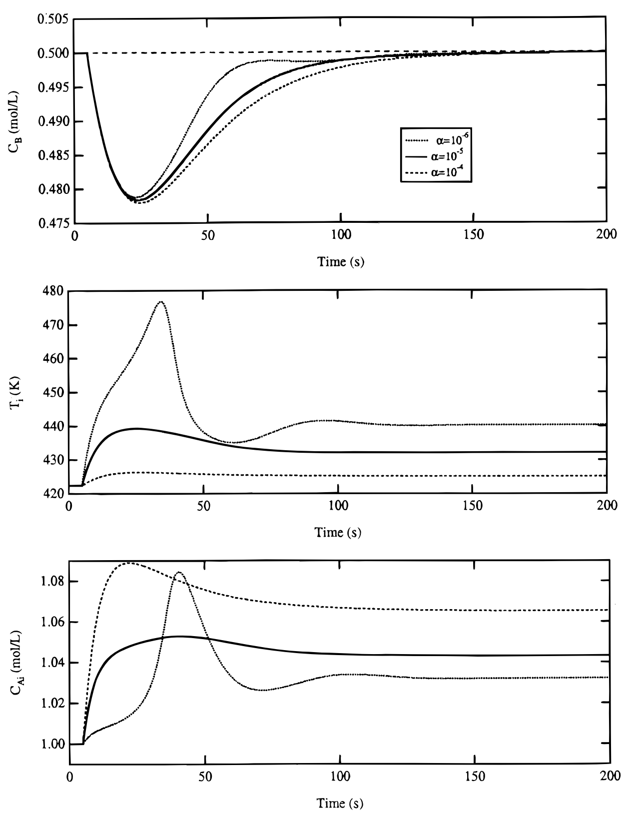

Figure 4. Effect of α on NHC performance (chemical reactor).

larger flow rate disturbances ($q = 1.33$ L/s). The effect of the NHC tuning parameter α on closed-loop performance for a flow rate disturbance ($q = 1.33$ L/s) is shown in Figure 4. As expected, the contribution of the second input C_{Ai} increases as α is increased. In this case, larger values of α yield improved disturbance rejection.

Table 2. Nominal Operating Conditions for Biochemical Reactor

symbol	definition	nominal value
D	dilution rate	0.04 h^{-1}
S_f	feed substrate concentration	20 g/L
$Y_{X/S}$	cell-mass yield	0.4 g/g
α	kinetic parameter	2.2 g/g
β	kinetic parameter	0.2 h^{-1}
μ_m	maximum specific growth rate	0.48 h^{-1}
P_m	product saturation constant	50 g/L
K_m	substrate saturation constant	1.2 g/L
K_i	substrate inhibition constant	22 g/L
X	biomass concentration	6.09 g/L
S	substrate concentration	4.76 g/L
P	product concentration	43.9 g/L

Biochemical Reactor. Now we apply the habituating control strategy to the following biochemical reactor model (Henson and Seborg, 1992):

$$\dot{X} = -DX + \mu X$$

$$\dot{S} = D(S_f - S) - \frac{1}{Y_{X/S}}\mu X \qquad (32)$$

$$\dot{P} = -DP + (\alpha\mu + \beta)X$$

where the growth rate μ is

$$\mu = \frac{\mu_m(1 - P/P_m)S}{K_m + S + S^2/K_i} \qquad (33)$$

Symbol definitions and nominal operating conditions are given in Table 2. A common choice for the manipulated input (u_1) and controlled output (y) are the dilution rate D and the substrate concentration S, respectively. These variables yield a well-defined relative degree $r_1 = 1$, but the linearized zero dynamics associated with u_1 are unstable at the operating point in Table 2. As a result, stable input–output linearization cannot be achieved using D as the only input variable. An internally stable closed-loop system can be obtained by employing the feed substrate concentration S_f as an additional manipulated input. However, large variations in S_f are undesirable in some applications.

We compare a nonlinear habituating controller that manipulates both D and S_f and an input–output linearizing controller that manipulates only D. The dilution rate is constrained as $0 \leq D \leq 0.1$ h^{-1}. To obtain the control affine form (1), the second input for NHC design is defined as $u_2 = DS_f$. It is easy to show that this input has relative degree $r_2 = 1$ and stable linearized zero dynamics. Both controllers utilize an input v that is designed to yield the closed-loop characteristic polynomial $(\epsilon s + 1)^2 = 0$, where $\epsilon = 3$ h. NHC employs target values that correspond to the operating conditions in Table 2: $\bar{D} = 0.04$ h^{-1}, $\bar{S}_f = 20$ g/L.

Theorem 1 can be used to determine the range of α values that yield a locally stable closed-loop system. As mentioned above, the linearized zero dynamics associated with D are unstable when S_f is not used. However, the stability of zero dynamics associated with D is changed dramatically when S_f is introduced as an additional input. It can be shown that the linearized zero dynamics actually are *stable* in this case. This seemingly anomalous result is attributable to the definition of the second input as $u_2 = DS_f$. Consequently, Theorem 1 shows that NHC provides local stability when $0 < \alpha < \infty$; we choose $\alpha = 20$.

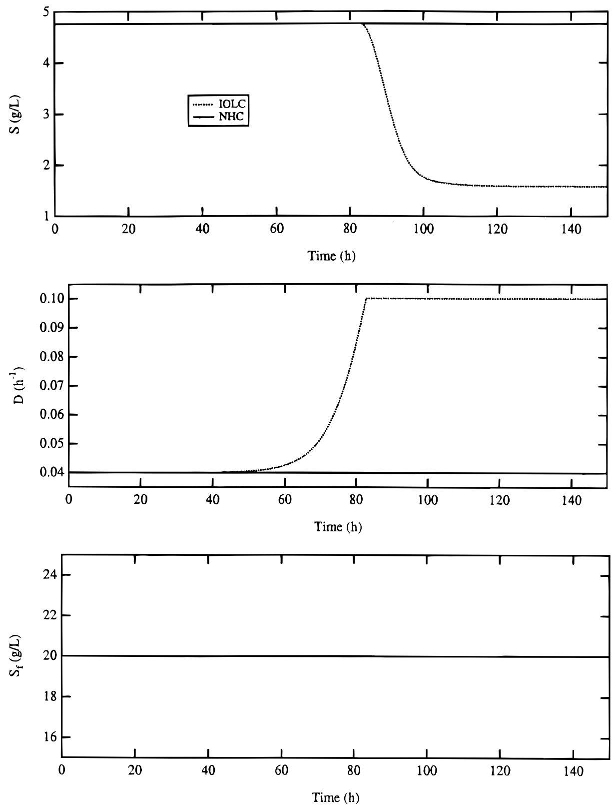

Figure 5. IOLC and NHC for stabilization at the nominal operating point (biochemical reactor).

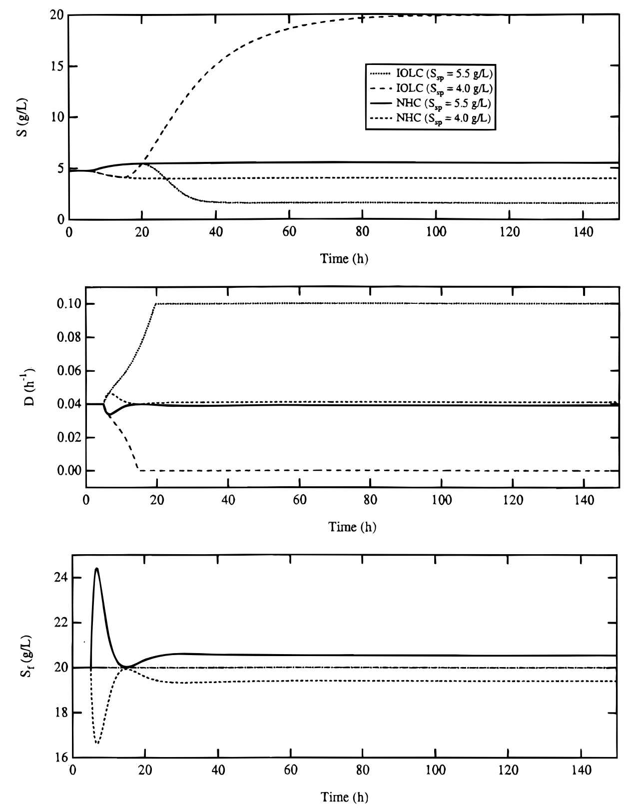

Figure 6. IOLC and NHC for setpoint changes (biochemical reactor).

Figure 5 compares the two controllers for operation at the nominal operating point in the presence of a very small initial condition error. IOLC is unable to stabilize the system as a result of the unstable zero dynamics, and D saturates at the upper constraint. When the constraint is removed, D is increased such that the system moves into the minimum phase region. The new steady state corresponds to the desired substrate concentration, but the product concentration (19.1 g/L) is much less than the value in Table 2. This behavior is undesirable in applications where a low dilution rate and/or high product concentration are desired. Figure 6 compares the controllers for positive (5.5 g/L) and negative (4 g/L) step changes in the substrate setpoint. IOLC cannot handle either setpoint change. The positive setpoint change causes D to saturate at the upper constraint, while the negative change causes saturation at the lower constraint. When the constraints are removed, the positive change can be handled as the system moves into the minimum phase region. However, the closed-loop system is unstable for the negative change. By contrast, NHC provides effective tracking of both setpoint changes by utilizing S_f as a second manipulated input.

The controllers are compared for a sudden change in the cell-mass yield ($Y_{X/S} = 0.45$ g/g) in Figure 7. Due to the unstable zero dynamics, IOLC cannot handle the disturbance as D saturates at the lower constraint. The system moves into the minimum phase region when the constraint is removed. NHC provides excellent disturbance rejection by using both D and S_f. Figure 8 shows the effect of the NHC tuning parameter α for the same parameter change. The disturbance rejection performance is comparable for all three values of α. Note that the utilization of *both* D and S_f is increased as α is decreased. This behavior can be explained by consider-

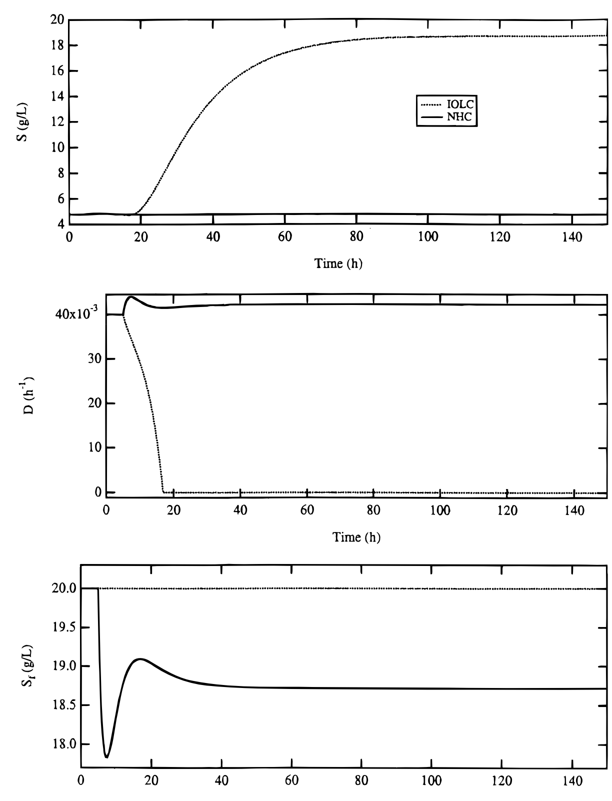

Figure 7. IOLC and NHC for a sudden change in the cell-mass yield (biochemical reactor).

ing the NHC cost function (7). In this example, a control affine system is obtained by defining the second input as $u_2 = DS_f$. Unexpected control moves are observed because DS_f is penalized rather than S_f.

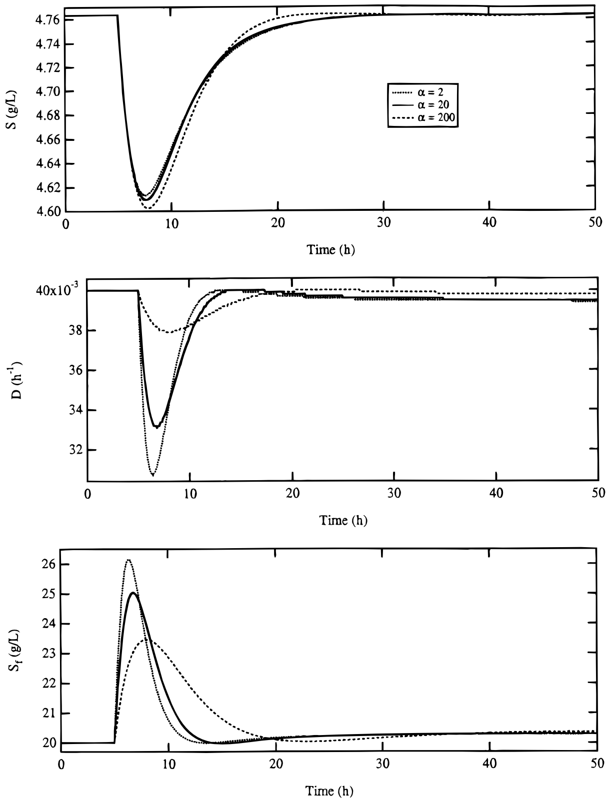

Figure 8. effect of α on NHC performance (biochemical reactor).

6. Summary and Conclusions

By emulating a control strategy used in biological systems, we have developed a controller design technique for nonlinear processes with more manipulated inputs than controlled outputs. The motivation for *habituating control* is that improved closed-loop performance can be achieved if all the available inputs are utilized. The nonlinear controller provides input–output linearization while simultaneously minimizing the cost of affecting control. In the single-output case, we have shown that the proposed method can provide a simple means to overcome the singularity and nonminimum phase problems. An extension of the controller design strategy for multiple-output processes also has been presented. The habituating control technique was successfully applied to nonlinear chemical and biochemical reactor models.

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Appendix

Proof of Theorem 1. First we consider the Jacobian linearization (19) of the extended nonlinear system assuming that the two inputs have equal relative degree. The associated transfer function model (20) allows a convenient representation of the linear plant operator in observability canonical state-space form:

$$\dot{z} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\alpha_0 \\ 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\alpha_1 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & -\alpha_{n-r} \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & -\alpha_{n-r-1} \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\alpha_{n-2} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1 & -\alpha_{n-1} \end{pmatrix} z + \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-r} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_{n-r} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} u_2 \quad (34)$$

$$y = (0 \quad \ldots \quad 0 \quad 1) z$$

Corresponding to this state-space model, we introduce the following definition of $\tilde{A}$, $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{c}$:

$$\dot{z} = \tilde{A}z + \tilde{b}_1 u_1 + \tilde{b}_2 u_2 \quad (35)$$
$$y = \tilde{c}z$$

It is shown by Isidori (1989) that one can construct an invertible transformation,

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{pmatrix} = Tx = \begin{pmatrix} \tilde{c}z \\ \tilde{c}\tilde{A}z \\ \tilde{c}\tilde{A}^2 z \\ \vdots \\ \tilde{c}\tilde{A}^{r-1} z \\ z_1 \\ \vdots \\ z_{n-r} \end{pmatrix} \quad (36)$$

which partitions the state vector into observable (ξ) and unobservable (η) variables. Internal stability of the input–output linearized system requires that the unobservable dynamics are stable. The unforced subsystem of unobservable state variables, also known as the zero dynamics, is obtained by setting $\xi = 0$:

$$\dot{\eta} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}\eta + \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-r-2} \\ \beta_{n-r-1} \end{pmatrix} u_1 + \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_{n-r-2} \\ \hat{\beta}_{n-r-1} \end{pmatrix} u_2 \quad (37)$$

The Jacobian linearization of the habituating controller (10)−(11) is given by:

$$u_1 = \frac{\beta_{n-r}}{\beta_{n-r}^2 + \alpha\hat{\beta}_{n-r}^2}(v - z_{n-r}) \quad (38)$$

$$u_2 = \frac{\alpha\hat{\beta}_{n-r}}{\beta_{n-r}^2 + \alpha\hat{\beta}_{n-r}^2}(v - z_{n-r})$$

When these relations are substituted into (37), the following result is obtained:

$$\dot{\eta} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -\dfrac{\beta_0\beta_{n-r} + \alpha\hat{\beta}_0\hat{\beta}_{n-r}}{\beta_{n-r}^2 + \alpha\hat{\beta}_{n-r}^2} \\ 1 & 0 & 0 & \cdots & 0 & 0 & -\dfrac{\beta_1\beta_{n-r} + \alpha\hat{\beta}_1\hat{\beta}_{n-r}}{\beta_{n-r}^2 + \alpha\hat{\beta}_{n-r}^2} \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & -\dfrac{\beta_{n-r-2}\beta_{n-r} + \alpha\hat{\beta}_{n-r-2}\hat{\beta}_{n-r}}{\beta_{n-r}^2 + \alpha\hat{\beta}_{n-r}^2} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\dfrac{\beta_{n-r-1}\beta_{n-r} + \alpha\hat{\beta}_{n-r-1}\hat{\beta}_{n-r}}{\beta_{n-r}^2 + \alpha\hat{\beta}_{n-r}^2} \end{pmatrix}\eta \quad (39)$$

It is straightforward to show that the stability of this subsystem is equivalent to the condition stated in the theorem; that is, the polynomial

$$\alpha\hat{\beta}_{n-r}\hat{N}(s) + \beta_{n-r}N(s) \quad (40)$$

is Hurwitz.

Now consider the case where the two inputs have different relative degrees. The transfer function model (20) can be manipulated such that the individual transfer functions have a common denominator:

$$y(s) = \frac{N(s)\, w_1(s) + s^{\mu}\hat{N}(s)\, w_2(s)}{s^{\mu}\, D(s)} \equiv \frac{\tilde{N}_1(s)\, w_1(s) + \tilde{N}_2(s)\, w_2(s)}{\tilde{D}(s)} \quad (41)$$

This model has the same form as that in the equal relative degree case. Recall that the final result hinged upon the stability of a weighted version of the individual numerator polynomials. In this case the numerator polynomial for w_2 is modified as shown, and the result in the theorem is obtained.

Proof of Corollary 2. Let $N(s)$ be the numerator polynomial associated with u_1 and $\hat{N}(s)$ be the numerator polynomial for u_2:

$$N(s) = \beta_{n-r_1}s^{n-r_1} + \beta_{n-r_1-1}s^{n-r_1-1} + \ldots + \beta_1 s + \beta_0$$

$$\hat{N}(s) = \hat{\beta}_{n-r_2}s^{n-r_2} + \hat{\beta}_{n-r_2-1}s^{n-r_2-1} + \ldots + \hat{\beta}_1 s + \hat{\beta}_0$$

We want to check the roots of the characteristic equation:

$$\beta_{n-r_1}N(s) + \alpha\hat{\beta}_{n-r_2}s^{r_2-r_1}\hat{N}(s) \equiv M(s) \qquad \alpha \geq 0$$

By assumption, the input u_1 is nonminimum phase and has a lesser relative degree than u_2 (*i.e.*, $N(s)$ contains right-half plane roots and $r_2 - r_1 > 0$). We define two real-valued functions over the polynomials:

$\{LC(p(x))\colon\ F[x] \rightarrow R\} \equiv$ coefficient of the lowest degree of x in $p(x)$

$\{HC(p(x))\colon\ F[x] \rightarrow R\} \equiv$ coefficient of the highest degree of x in $p(x)$

Note the following:

$$LC(M(s)) = \beta_{n-r_1}\beta_0$$

$$HC(M(s)) = \beta_{n-r_1}^2 + \alpha\beta_{n-r_2}^2 > 0 \qquad \forall\ \alpha \geq 0$$

Thus, to prove the corollary, it is sufficient to show that $\beta_{n-r_1}\beta_0 \leq 0$ for an odd number of right-half plane (RHP) zeros since the first criterion of the Routh−Hurwitz test fails in this case.

First note that $\beta_{n-r_1} \neq 0$. If $\beta_{n-r_1}\beta_0 = 0$, then $\beta_0 = 0$, which implies that $M(s)$ has a root at the origin for any value of the tuning parameter α and any number of additional closed RHP roots. This implies $M(s)$ is not Hurwitz.

Now assume $\beta_0 \neq 0$ and write $N(s) = Q(s)\, P(s)$, where $Q(s)$ contains all the open left-half plane (LHP) roots of $N(s)$ and $P(s)$ contains all the closed right-half plane (RHP) roots. Since $Q(s)$ has only negative roots, the following must be true:

$$HC(Q(s))\cdot LC(Q(s)) > 0$$

Further partition $P(s) = P_c(s)\, P_r(s)$, where $P_c(s)$ contains all the closed RHP complex roots and $P_r(s)$ contains all the RHP real roots. First we consider the contributions from $P_c(s)$,

$$P_c(s) = (a_1s^2 + b_1s + c_1)(a_2s^2 + b_2s + c_2)\ \ldots\ (a_ls^2 + b_ls + c_l)$$

where $a_ic_i > 0$ and $a_ib_i \leq 0$. It is clear that the following holds:

$$HC(P_c(s))\cdot LC(P_c(s)) = (a_1a_2\ \ldots\ a_l)(c_1c_2\ \ldots\ c_l) = (a_1c_1)(a_2c_2)\ \ldots\ (a_lc_l) > 0$$

Now we consider the contribution of $P_r(s)$,

$$P_r(s) = (a_1s + b_1)(a_2s + b_2)\ \ldots\ (a_ps + b_p)$$

where $a_i, b_i \in R$ are such that $a_i b_i < 0$. If p is even and if an odd number of $a_i < 0$, then an odd number of $b_i < 0$. Similarly, if p is even and an even number of $a_i < 0$, then an even number of $b_i < 0$. This shows that

$$HC(P_r(s)) < 0 \Rightarrow LC(P_r(s)) < 0$$

$$HC(P_r(s)) > 0 \Rightarrow LC(P_r(s)) > 0$$

which implies that $HC(P_r(s)){\cdot}LC(P_r(s)) > 0$ if there is an even number of real roots in $P_r(s)$. If p is odd then,

$$P_r(s) = (\gamma s + \delta)(a_{p-1}s^{p-1} + ... + a_0)$$

where $\gamma\delta < 0$ and $a_{p-1}a_0 > 0$. Thus, if p is odd:

$$HC(P_r(s)){\cdot}LC(P_r(s)) = (\gamma a_{p-1})(\delta a_0) = (\gamma\delta)(a_{p-1}a_0) < 0$$

We now can combine all the contributions to $N(s)$:

$$\begin{aligned}\beta_{n-r_1}\beta_0 &= HC(N(s)){\cdot}LC(N(s)) \\ &= HC(Q(s)){\cdot}HC(P_c(s)){\cdot}HC(P_r(s)){\cdot}LC(Q(s)){\cdot} \\ &\qquad LC(P_c(s)){\cdot}LC(P_r(s)) \\ &= [HC(Q(s)){\cdot}LC(Q(s))][HC(P_c(s)){\cdot} \\ &\qquad LC(P_c(s))][HC(P_r(s)){\cdot}LC(P_r(s))]\end{aligned}$$

Thus,

$$\beta_{n-r_1}\beta_0 \begin{cases} < 0 & \text{if } N(s) \text{ contains an odd number of RHP roots} \\ > 0 & \text{if } N(s) \text{ contains an even number of RHP roots} \end{cases}$$

and the proof is complete.

Proof of Theorem 2. The set of equations (28) has a unique solution if the $m \times m$ matrix

$$\begin{bmatrix} A_1(x) & A_2(x) \\ -(A_1^{-1}(x)\, A_2(x))^{\mathrm{T}}\boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_2 \end{bmatrix} \tag{42}$$

has full rank m at x_0. For simplicity, we omit the state dependence of the matrices. Because A_1 is invertible by assumption, the first set of equations in (28) can be premultiplied by A_1^{-1} to yield the matrix:

$$\begin{bmatrix} I_p & A_1^{-1}A_2 \\ -(A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_2 \end{bmatrix} \tag{43}$$

This matrix can be rewritten as follows after a simple row reduction operation:

$$\begin{bmatrix} I_p & A_1^{-1}A_2 \\ 0 & \boldsymbol{\Gamma}_2 + (A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1(A_1^{-1}A_2) \end{bmatrix} \tag{44}$$

Because of its block diagonal structure, this matrix is full rank, and therefore the equations (28) have a unique solution, if:

$$\mathrm{rank}[\boldsymbol{\Gamma}_2 + (A_1^{-1}A_2)^{\mathrm{T}}\boldsymbol{\Gamma}_1(A_1^{-1}A_2)] = m - p \tag{45}$$

It is easy to show that this condition holds if either of the two assumptions in the theorem are satisfied.

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