Frequency Response Analysis

Karl D. Hammond

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1 Introduction

Frequency Response (sometimes called FR) is a key analysis tool for control of some dynamic systems. This analysis is based on the fact that if the input to a stable process is oscillated at a frequency ω, the long-time output from the process will also oscillate at a frequency ω, though with a different amplitude and phase.

Advanced control techniques related to frequency response are crucial to the performance of devices in the communications industry and other facets of electrical engineering. Imagine what would happen, for example, if your cellular phone amplified all frequencies, including the high-frequency noise produced by air moving past the mouthpiece, at the same volume as your voice!

The frequency response technique can also be valuable to mechanical engineers studying things like airplane wing dynamics or chemical engineers studying diffusion or process dynamics.

As an example of what happens to a system with an oscillatory input, consider a system consisting of a damped harmonic oscillator (I've used symbols usually reserved for masses on springs here):

$$m\frac{d^2x}{dt^2} = -kx - \epsilon\frac{dx}{dt} + F, \tag{1}$$

where x is the deviation from the equilibrium position, t is time, ϵ is a friction coefficient, and F is the force applied to the spring (the input). The output is simply the measured position, so $y = x$. We will define the input (for convenience) as $u = F/m$, the force per unit mass. The transfer function from input to output (assuming zero initial conditions) is thus

$$P(s) = \frac{1}{s^2 + \frac{\epsilon}{m}s + \frac{k}{m}} = \frac{1}{s^2 + \zeta s + \omega_0^2}, \tag{2}$$

where $\zeta = \epsilon/m$ and $\omega_0 = \sqrt{k/m}$. If we put an oscillatory signal (such as a sine function) as input to this "plant," we get an expression for the output, $y(t)$. The Laplace transform of $u(t) = \sin(\omega t)$ is $U(s) = \frac{\omega}{s^2+\omega^2}$, so

$$Y(s) = P(s)U(s) = \frac{1}{s^2 + \zeta s + \omega_0^2}\frac{\omega}{s^2 + \omega^2}. \tag{3}$$

We can decompose this equation by partial fractions into the following form:

$$Y(s) = \frac{A}{s^2 + \zeta s + \omega_0^2} + \frac{Bs}{s^2 + \zeta s + \omega_0^2} + \frac{C\omega}{s^2 + \omega^2} + \frac{Ds}{s^2 + \omega^2}. \tag{4}$$

The output $y(t)$ is thus of the form

$$y(t) = \alpha_1 e^{-\beta_1 t} \sin(\gamma_1 t) + \alpha_2 e^{-\beta_2 t} \cos(\gamma_2 t) + C \sin(\omega t) + D \cos(\omega t), \tag{5}$$

where α_i, β_i, and γ_i are positive real constants. The first two terms eventually decay away, leaving the last two terms. These two terms oscillate with frequency ω, as expected from the note above. We can use some trigonometry to express these in a more obvious way:

$$\begin{aligned} \lim_{t\to\infty} y(t) = C \sin(\omega t) + D \cos(\omega t) &= M \left(\frac{C}{M} \sin(\omega t) + \frac{D}{M} \cos(\omega t) \right) \\ &= M \left(\cos\phi \sin(\omega t) + \sin\phi \cos(\omega t)\right) = M \sin(\omega t + \phi). \end{aligned} \tag{6}$$

The phase angle can thus be computed as $\phi = \arctan(D/C)$. The magnitude, M, and the phase, ϕ, are intimately linked with the nature of the transfer function and thus of the plant itself.

2 The Impulse Response

What, exactly does the function $P(s)$, the "plant transfer function," represent? That is, if we invert the Laplace transform and obtain the function $p(t) = \mathscr{L}^{-1}[P(s)]$, what does $p(t)$ represent? Remember that $P(s)$ may be a composite of several differential equations! However, inverting $P(s)$ by itself corresponds to the response due to $U(s) = 1$; we now need only to find the function $u(t)$ that transforms to this in the Laplace domain.

2.1 Delta Functions

There are two kinds of so-called delta functions that will be of use to use here. The first is the Kronecker delta, defined such that

$$\sum_{n=-\infty}^{\infty} \delta[n-m] a_n = a_m. \tag{7}$$

The Kronecker delta can be written as δ_{nm} instead of $\delta[n-m]$. Either way, it is true that

$$\delta[n-m] = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{8}$$

This function serves to "pull out" the value of the coefficient a in the series a_n corresponding to m. This works well for *discrete* systems, but this delta function is useless for *continuous* functions, where integrals instead of sums are involved.

Another type of delta function, introduced by British mathematician Paul Dirac in the 1920's, is called the Dirac delta, defined—by analogy to Equation (7)—such that

$$\int_{-\infty}^{\infty} \delta(t-\tau) f(t) dt = f(\tau) \tag{9}$$

Thus the Dirac delta function serves the same purpose for continuous functions as its partner the Kronecker delta serves for discrete functions: it "pulls out" the value of the function at that point in the domain.

The Dirac delta function has a few properties similar to those of the Kronecker delta. First, we can immediately see from Eq. (9) that

$$\int_{-\infty}^{\infty} \delta(t) dt = 1. \tag{10}$$

Second, the function has the property that

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0, \end{cases} \tag{11}$$

meaning that this function can be envisioned as the limit of the Kronecker delta function as the size of the discrete unit approaches zero, preserving the area of each discrete cell.

2.2 Laplace Transform of the Dirac Delta Function

By the property in Eq. (9), we immediately see that the Laplace transform of the delta function is

$$\mathscr{L}[\delta(t-\tau)] = e^{-\tau s}, \tag{12}$$

meaning that $\mathscr{L}[\delta(t)] = 1$. From our discussion earlier, this means that the input that produces a transformed input $U(s) = 1$ is none other than $u(t) = \delta(t)$. In this context, the Dirac delta function is called the *unit impulse* function: it can be modeled as a sudden burst of input with integral 1 at time zero. The output $p(t)$ resulting from a delta function input is thus termed the *impulse response*. This means that **the transfer function $\boldsymbol{P(s)}$ is the Laplace transform of the impulse response, $\boldsymbol{p(t)}$**.

At this point, it is important to note that multiplication of transfer functions in the Laplace domain corresponds to *convolution* of functions in the time domain, where the convolution of two functions f and g is defined as

$$f \circ g = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t f(\tau) g(t-\tau) d\tau. \tag{13}$$

This integral is the "generalized multiplication" in the time domain, and it works out that

$$\mathscr{L}[f \circ g] = F(s) G(s). \tag{14}$$

It is therefore *not* true that $y(t) = g(t)u(t)$; it *is* true that $Y(s) = G(s)U(s)$.

Equation (13) combined with Eq. (12) provides a simple way to derive the Laplace transform of a function with a time delay: rewrite the function as the convolution of a Delta function and the un-delayed function via Eq. (9), then take the transform of each function and multiply.

3 Aside: Fourier Series and Fourier Transforms

3.1 Fourier Series

Fourier series are a method to express any function that is continuous on a given interval as a sum of of sines and cosines on that interval. Fourier supposed, and later proved, that if a function $f(x)$ is continuous on the interval $(0, L)$, then one can write

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \sin(k_n x) + b_n \cos(k_n x),$$

where $k_n = \frac{n\pi}{L}$; this makes for the following properties:

- $\int_{-\infty}^{\infty} \sin(k_n x)\sin(k_m x)dx = \frac{L}{2}\delta[n-m]$
- $\int_{-\infty}^{\infty} \cos(k_n x)\cos(k_m x)dx = \frac{L}{2}\delta[n-m]$
- $\int_{-\infty}^{\infty} \sin(k_n x)\cos(k_m x)dx = 0.$

Therefore if we multiply both sides by $\sin(m\pi x/L)$ or $\cos(m\pi x/L)$ and integrate, we obtain

$$a_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right) \tag{15a}$$

$$b_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right) \tag{15b}$$

Some people call Eq. (15) the definition of the *Finite Fourier Transform*[1] or the *Discrete Fourier Transform*.

3.2 The Fourier Transform

The purpose of the Fourier transform is similar to that of the Laplace transform in that it seeks to transform a differential equation into an algebraic equation. However, whereas the Laplace transform uses the complex variable s (which has units of inverse time but is otherwise difficult to interpret), the Fourier transform uses complex waves of the from e^{ikx} or some variation on that form to write the transformed function as a sum of waves of all frequencies (represented here by the frequency ν or wavenumber $\bar{\nu}$, or their angular equivalents ω and k).

[1] This is occasionally, and somewhat awkwardly, referred to as the FFT by some authors. That term is (almost) universally reserved for the Fast Fourier Transform, an algorithm used to *take* discrete Fourier transforms. . .

The Fourier Transform can be defined by several integral pairs, depending on the units of the quantity to be transformed and the desired units of the transformed variable. If f is a function of displacement x, with units of length, then the resulting variable can be defined to have units of wavenumber ($\bar{\nu}$) or angular wavenumber (k). In wavenumber units, we can define[2]

$$\mathscr{F}[f(x)] = F(\bar{\nu}) = \int_{-\infty}^{\infty} e^{-2\pi i\bar{\nu}x} f(x)dx \tag{16a}$$

$$\mathscr{F}^{-1}[F(\bar{\nu})] = f(x) = \int_{-\infty}^{\infty} e^{2\pi i\bar{\nu}x} F(\bar{\nu})d\bar{\nu}. \tag{16b}$$

In this form, x has units of length, and $\bar{\nu}$ has units of inverse length (wavenumbers). A similar form which is more common is

$$\mathscr{F}[f(x)] = F(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx \tag{17a}$$

$$\mathscr{F}^{-1}[F(k)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} F(k)dk, \tag{17b}$$

where k is an *angular* wavenumber (radians/unit length). The factor of 2π is sometimes moved to the forward transform; the only important part is that the entire transform have a factor of $1/2\pi$ somewhere in this form. In fact, some authors (especially physicists), define the transform with these units as

$$\mathscr{F}[f(x)] = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx \tag{18a}$$

$$\mathscr{F}^{-1}[F(k)] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} F(k)dk. \tag{18b}$$

Which you use is largely a matter of preference, provided you remember that the factor of 2π is to cancel the units of k, which is in radians per unit length. The form defined by Eq. (18) would therefore have a transformed function with the somewhat awkward units of radians$^{1/2}$.

If the function in question is a function of time (instead of distance), then two other forms come into play. They are analogous to the spatial forms, except that the transformed variable has units of frequency instead of wavenumber. For frequencies in Hz (cycles/second) and analogous units:

$$\mathscr{F}[f(t)] = F(\nu) = \int_{-\infty}^{\infty} e^{-2\pi i\nu t} f(t)dt \tag{19a}$$

$$\mathscr{F}^{-1}[F(\nu)] = f(t) = \int_{-\infty}^{\infty} e^{2\pi i\nu t} F(\nu)d\nu. \tag{19b}$$

[2]Note that each of these transforms can be replaced by its complex conjugate with impunity. That is, the forward transform can have $-i$ *or* i in the exponent, provided the reverse transform has the opposite sign. The factor of $1/2\pi$ can also be moved between the forward and reverse transforms, depending on the desired units of the transformed function.

Similarly, for angular frequencies:

$$\mathscr{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \tag{20a}$$

$$\mathscr{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} F(\omega) d\omega. \tag{20b}$$

We will define Equations (20) as the definition of the Fourier Transform, $\mathscr{F}[f(t)]$, for elements in the time domain.

4 Frequency from the Transfer Function

Recall that the open-loop transfer function $G(s) = P(s)C(s)$, where $C(s)$ is the transfer function of the controller (from error to input, as it were). Also recall (Section 2) that the transfer function is the Laplace transform of the impulse response. The transfer function can thus be found by the definition of the Laplace transform:

$$G(s) = \mathscr{L}[g(t)] = \int_{0}^{\infty} e^{-st} g(t) dt. \tag{21}$$

Strictly speaking, this is the *unilateral* Laplace transform; the *bilateral* Laplace transform is integrated over the entire real axis. Since outputs are assumed to be at steady-state (or at least unknown) prior to time $t = 0$, these are equivalent for our purposes. If we restrict s to be purely imaginary,[3] we can let $s = i\omega$ and Eq. (21) becomes

$$G(i\omega) = \int_{0}^{\infty} e^{-i\omega t} g(t) dt = \mathscr{F}[g(t)]. \tag{22}$$

Comparing this to Eq. (20a) leads us to the equality on the right (recalling that $g(t) = 0$ for $t < 0$). In short, $G(i\omega)$ is the Fourier transform of the impulse response: the value of $G(i\omega)$ at each value of ω represents the contribution of a wave of frequency ω to the value of $g(t)$. If we eliminate the contributions from all other waves (say, by exciting at that frequency and allowing the other states to relax away), the Fourier transform tells us the final response of the system to that oscillation, including its magnitude and phase-shift.

To obtain the magnitude and phase directly from the transfer function $G(s)$ (the open-loop transfer function), we separate $G(i\omega)$ into real and imaginary parts:[4] (see Seborg et al., page 337–338)

$$G(i\omega) = R(\omega) + iI(\omega).$$

[3] I mean this in the mathematical sense, not the metaphysical sense, of course.

[4] Hint: multiply the numerator and denominator by the complex conjugate of the denominator.

The amplitude ratio (AR) and phase lag (ϕ) of the long-time oscillations are then

$$\text{AR} = |G(i\omega)| = \sqrt{R^2 + I^2} \tag{23}$$

$$\phi = \angle G(i\omega) = \arctan\left(\frac{I}{R}\right). \tag{24}$$

Note that the magnitude of the actual oscillation will be $a\text{AR}$, where a is the amplitude of the exciting oscillation.

5 Analysis Tools

5.1 Bode Plots

Named for Henrik Bode (bōd′ə, but everyone says bōd′ē), the *Bode diagram* is one possible representation of the long-time modulus and phase of the output in response to a sinusoidal input. It consists of two stacked plots with frequency[5] on the horizontal axis.

The upper plot is a log-log plot of magnitude against frequency. The magnitude is nearly always displayed in units of *decibels* (dB), such that

$$M \text{ (dB)} = 20 \log_{10} M.$$

Note that a decibel is *usually* defined in this way, though occasionally the factor in front is 10 instead of 20: this is because a decibel was originally invented to describe loss of *power* over telephone cables, and power is proportional to the *square* of amplitude (so the factor of 10 is for ratios of *power*, while the factor of 20 is for ratios of the *amplitude*). It is worth checking which definition they are using if you find a paper in the literature.

The lower plot is a log-linear plot of phase *lead* against frequency. The phase is typically plotted in degrees. The phase lead is always negative (the output lags behind the error signal as opposed to leading it).

Examples of Bode diagrams are shown in Figures 1–6.

The Bode plot is very useful for determining various aspects of a process. For example, it is desirable in most processes to have good disturbance rejection at high frequencies (that is, the process is not affected overly much if the signal is noisy). This corresponds to a fall-off in magnitude to the right of the Bode plot. The phase plot gives information about the *order* of the process (if such information is unknown): the general rule is, the phase will shift $-90°$ for each first-order term in the plant and/or controller. Thus a third-order system would exhibit a phase shift of $-270°$ at high frequencies. At intermediate frequencies, however, the decay times of each of the individual first-order terms may or may not be significant, so the phase lag usually follows a "step" pattern, with the number of steps being equal to the order of the system.

[5] That's *angular* frequency!

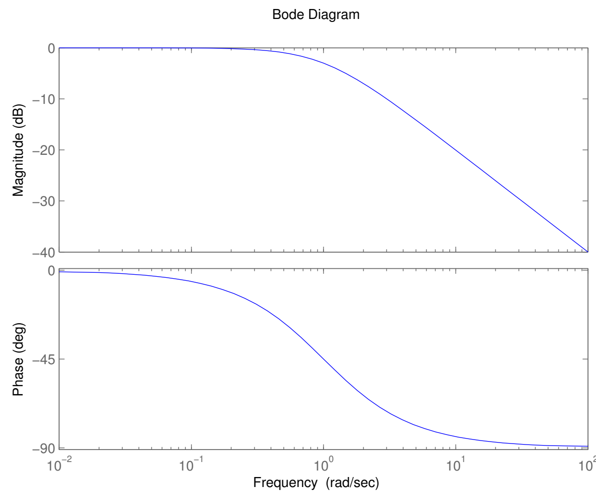

Figure 1: Open-loop Bode plot for a first-order system defined by the transfer function $G(s) = \frac{1}{s+1}$.

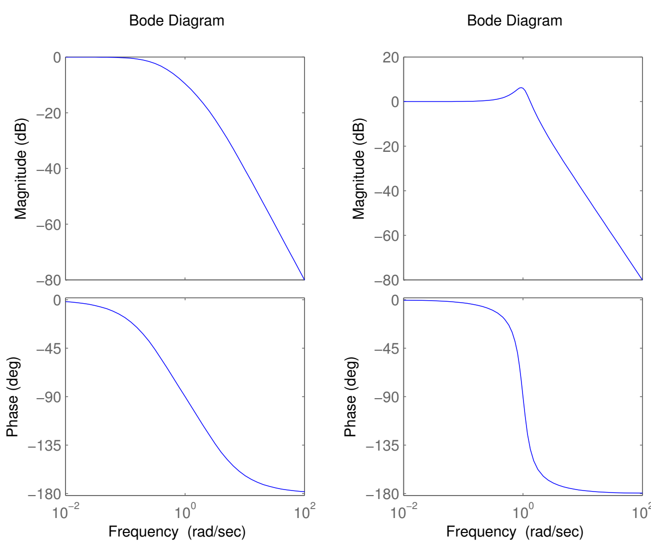

Figure 2: Open-loop Bode plots for second-order systems defined by the transfer function $G(s) = \frac{1}{s^2+2\zeta s+1}$. *Left*: Overdamped ($\zeta = 3$). *Right*: underdamped ($\zeta = 0.25$). The maximum on the underdamped plot corresponds to the *resonance frequency* of this second-order system.

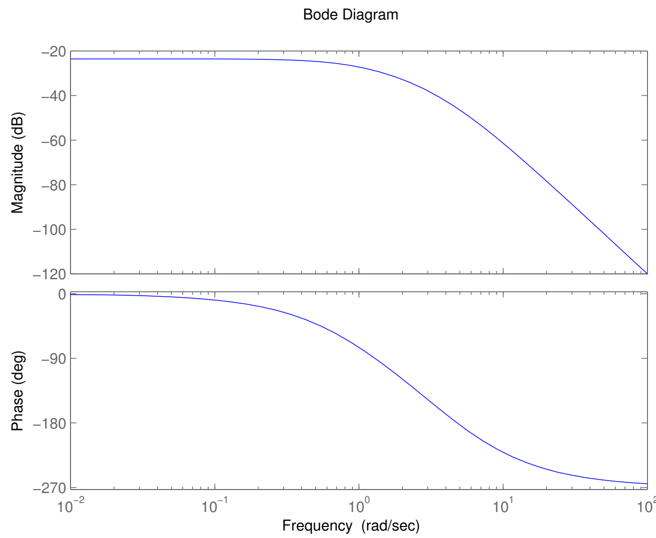

Figure 3: Open-loop Bode plot for the third-order system defined by the transfer function $G(s) = \frac{1}{(s+5)(s+3)(s+1)}$.

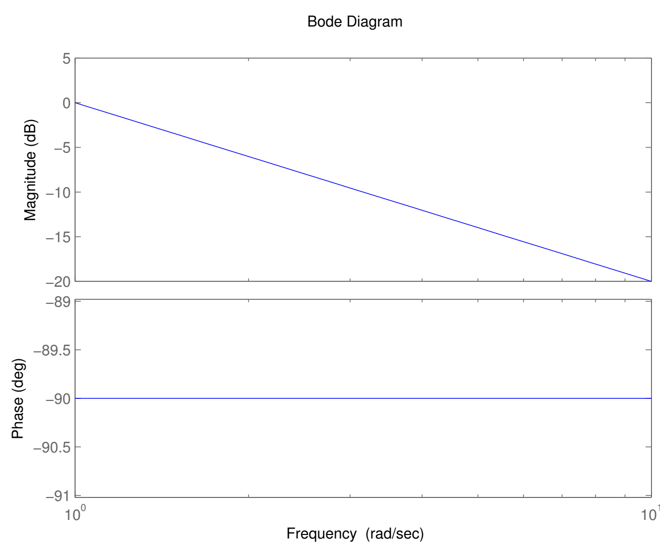

Figure 4: Open-loop Bode plot for an integrating controller element.

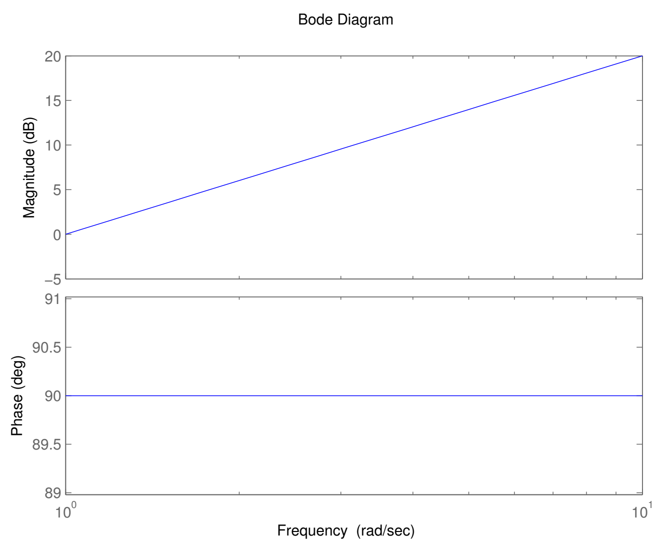

Figure 5: Open-loop Bode plot for a differentiating controller element.

6 Bode Diagrams for Closed-Loop Processes

Usually Bode diagrams are constructed for open-loop processes—this gives information about the relative gains and such. For processes, however, it is occassionally necessary to resort to constructing the diagram on the *closed*-loop process. In this case, you oscillate not the error but the set point.

If the plant, $P(s)$, is described by the ratio of two polynomials in s, then $P(s) = P_{n-1}(s)/P_n(s)$. The transfer function from the set point, r, to the output, y, is

$$H_{yr} = \frac{C(s)P_{n-1}(s)}{P_n(s)} \frac{1}{1+P(s)C(s)} = \frac{C(s)P_{n-1}(s)}{P_n(s)+C(s)P_{n-1}(s)} \tag{25}$$

If the controller is a simple PI or PID controller (with integrator on), then $C(s) = K_c(1 + 1/\tau_i s + \tau_d s)$. Multiplying the top and bottom of the closed-loop transfer function by $\tau_i s$ yields

$$\begin{aligned} H_{yr} &= \frac{K_c(1 + 1/\tau_i s + \tau_d s)P_{n-1}(s)}{P_n(s) + K_c(1 + 1/\tau_i s + \tau_d s)P_{n-1}(s)} \\ &= \frac{K_c(1 + \tau_i s + \tau_i \tau_d s^2)P_{n-1}(s)}{\tau_i s P_n(s) + K_c(1 + \tau_i s + \tau_i \tau_d s^2)P_{n-1}(s)} \end{aligned} \tag{26}$$

The apparent order of the process is now $n + 1$. However, the right choice of gains can cause very, very odd behavior in even simple plants, as can be seen in Figure 6. The errant behavior is due to the presence of zeros that were not present in the open-loop transfer function that are artifacts of the integrator and differentiator.

6.1 Nyquist Plots

Nyquist plots provide an alternative to Bode diagrams in which the frequency does not appear explictly, instead being a polar plot of magnitude and phase in which the frequency is an implicit parameter. The Nyquist diagram is simpler than the Bode diagram, and is often used for multiple loop control schemes (such as cascade control) for that reason.

The idea of the Nyquist plot is to plot $G(i\omega)$ in polar coordinates with $r = |G(i\omega)|$ and $\theta = \angle G(i\omega)$. Another equivalent method is to plot $\Re[G(i\omega)]$ on the horizontal axis and $\Im[G(i\omega)]$ on the vertical axis. Note that, unlike Bode plots, Nyquist plots *must* be plotted using the *open-loop* transfer functions in order to provide any sort of useful information.

The Nyquist plot provides an immediate determination of loop stability for proposed interacting loops, for example. These conditions are discussed in the next section. Some examples of Nyquist plots are shown in Figures 7–9.

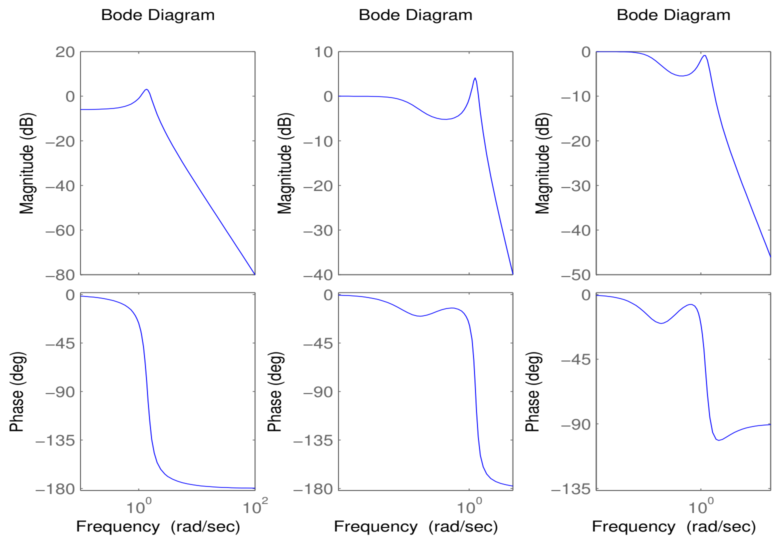

Figure 6: Closed-loop Bode plots for a second-order plant with a P, PI, and PID controller, respectively. The plant is given by $P(s) = \frac{1}{s^2+0.5s+1}$. The gains are $K_p = 1$, $\tau_i = 10$ s, and $\tau_d = 0.5$ s.

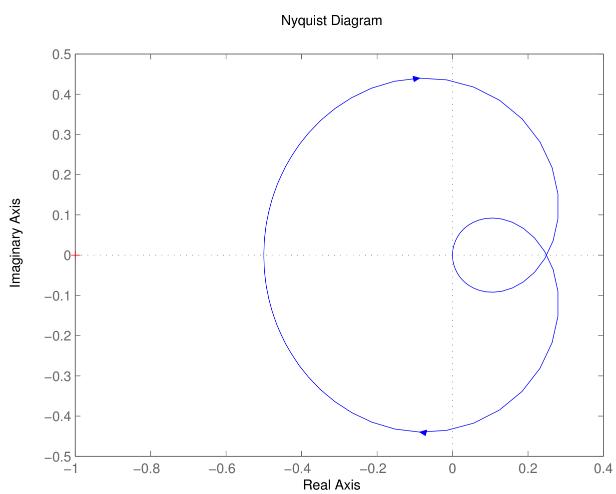

Figure 7: Nyquist plot for a second-order stable system defined by the transfer function $G(s) = 0.5\frac{s-1}{s^2+2s+1}$.

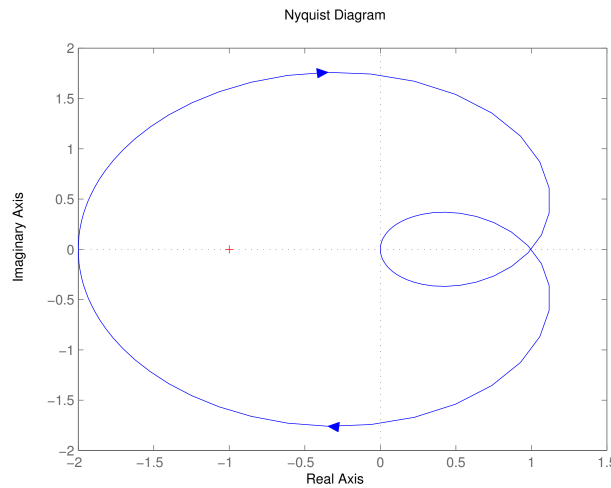

Figure 8: Nyquist plot for a second-order unstable system defined by the transfer function $G(s) = 2\frac{s-1}{s^2+2s+1}$.

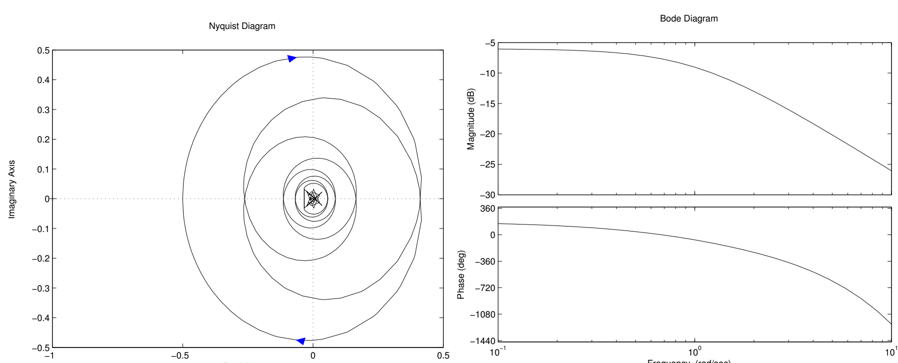

Figure 9: Nyquist plot and corresponding Bode plot for a second-order stable system defined by the transfer function $G(s) = 0.5e^{-2s}\frac{s-1}{s^2+2s+1}$. The time delay actually creates a spiral inward (which MATLAB does a bad job of rendering due to discretization).

6.2 Stability Criteria

The analysis diagrams presented in this section can be used to determine if a proposed control scheme will be asymptotically stable. These methods are generally preferable to other criteria such as the Routh criterion (see Seborg et al., Chapter 11), as they handle time delays exactly and provide an estimate of the relative stability of a process (how stable the system is, not just whether it is stable).

Gain and Phase Margins The idea with the Bode criterion is to find the roots of $1 + G(s) = 0$, the characteristic equation. Each root must therefore satisfy $G(s) = -1 = e^{i\pi}$, which corresponds to a magnitude of 1 and a phase of $-180°$. We thus define a *crossover frequency* as being the frequency at which the magnitude of the transfer function is exactly 1 (0 dB). We also define the *critical frequency* as the frequency at which the phase lag is equal to 180°. The *gain margin* is defined as the inverse of the amplitude ratio at the critical frequency. The *phase margin* is 180° plus the phase angle at the crossover frequency. The gain margin is a measure of how much the overall open-loop gain can be increased before the process becomes unstable. The phase margin provides an indication of how much time delay the system can tolerate before becoming unstable. A guideline from Seborg et al.: "A well-tuned controller should have a gain margin between 1.7 and 4.0 and a phase margin between 30° and 45°."

Bode Stability Criterion The Bode stability criterion is thus: If $G(s)$ has more poles than zeros and no poles in the right half-plane (excluding the origin), and if $G(i\omega)$ has only one critical frequency ω_c and one crossover frequency, then the loop is stable if and only if $|G(i\omega_c)| < 1$. This is equivalent to saying the gain margin must be greater than 1 (i.e., 0 dB).

Nyquist Stability Criterion The idea with the Nyquist criterion is to find the number of times $1 + G(i\omega)$ circles the origin, equivalent to the number of times $G(i\omega)$ circles the point $(-1, 0)$, in the clockwise direction. This number, N, is the number of right half-plane roots of the denominator of the closed-loop transfer function (including effects due to time delays). Next, find the number of right half-plane poles of the open-loop transfer function, P, which will also be poles of the closed-loop transfer function. The total number of RHP poles of the closed-loop transfer function is always $Z = N + P = 0$ for a stable system.

The derivation of the Nyquist criterion is in Seborg et al. I have borrowed their notation for most of this writeup, since that is what you are familiar with.