Problem 1. Consider the following dynamic model for sustained oscillations in yeast glycolysis:

\[
\begin{align*}
\frac{dG}{dt} &= V_{in} - k_1 GA = f_1(G, A) \\
\frac{dA}{dt} &= 2k_1 GA - k_p \frac{A}{K_m + A} = f_2(G, A)
\end{align*}
\]

where $G$ and $A$ are the intracellular concentrations of glucose and ATP, respectively, $V_{in} = 0.36$ is the constant flux of glucose into the yeast cell, $k_1 = 0.02$ is an enzyme activity, and the parameters $k_p = 6.0$ and $K_m = 13.0$ determine the kinetics of ATP degradation.

1. Show that the given parameter values produce a single steady-state solution $\bar{G}$ and $\bar{A}$.

2. Show that the linearized model at this steady-state point has the form:

\[
\frac{dy}{dt} = Ay
\]

where $x = [G' \ A']^T$.

3. Determine the stability of the steady state by computing the eigenvalues of the $A$ matrix. Interpret your result.

Problem 2. Consider the batch operation of a flash drum for separating a binary liquid mixture. The drum has liquid molar holdup $M$ and liquid mole fraction of the more volatile component $x$. A constant rate of heat $Q$ is applied to the drum, producing a constant vapor molar flow rate $V$. The vapor mole fraction of the more volatile component is denoted $y$. A linear vapor-liquid equilibrium relation $y = kx$, where $k > 1$, is assumed. Mass balances yield the following ordinary differential equation system:

\[
\begin{align*}
\frac{dM}{dt} &= -V, \quad M(0) = M_0 \\
\frac{d(Mx)}{dt} &= -Vy, \quad x(0) = x_0
\end{align*}
\]

1. Solve the ODEs analytically to obtain:

\[
\begin{align*}
M(t) &= M_0 - Vt \\
x(t) &= x_0 \left( \frac{M_0 - Vt}{M_0} \right)^{k-1}
\end{align*}
\]
2. Linearize the ODEs about the initial conditions to generate linear ODEs for \( M'(t) = M(t) - M_0 \) and \( x'(t) = x(t) - x_0 \). Analytically solve the first linear ODE and formulate the second linear ODE to obtain:

\[
\begin{align*}
M'(t) &= -Vt \\
\frac{dx'}{dt} + \frac{V}{M_0}(k-1)x' &= -\frac{V}{M_0}(k-1)x_0 \left(1 + \frac{V}{M_0}t\right)
\end{align*}
\]

How would you solve the second ODE? How would you expect your solution to compare to the analytical solution in part 1?

3. Generate numerical solutions to the original ODEs given in part 1 by applying the forward Euler method with step size \( h \). Show that:

\[
\begin{align*}
M_n &= M_0 - nVh \\
x_{n+1} &= x_n - h \left(\frac{V}{M_0 - nVh}\right) (k-1)x_n
\end{align*}
\]

How would you expect your solutions to compare to the analytical solutions in part a?