Lake Models

- Completely mixed analytical solutions (can accommodate time variable loading)
  - Single lake (lecture 4 & 5)
  - Lakes in Series
- Compartmentalized models
  - Two layer models (analytical)
  - 1, 2 and 3 dimensional approaches (numerical)
**Completely mixed Lakes**

\[ k_1 V_1 c_1 \]

\[ k_2 V_2 c_2 \]

\[
V_1 \frac{dc_1}{dt} = W_1 - Q_{12} c_1 - k_1 V_1 c_1
\]

\[
V_2 \frac{dc_2}{dt} = W_2 + Q_{12} c_1 - Q_{23} c_2 - k_2 V_2 c_2
\]

**Steady state solution**

\[
W_1 = Q_{12} c_1 + k_1 V_1 c_1
\]

\[
W_2 = -Q_{12} c_1 + Q_{23} c_2 + k_2 V_2 c_2
\]

\[
c_1 = \frac{W_1}{Q_{12} + k_1 V_1}
\]

\[
c_2 = \frac{W_2 + Q_{12} c_1}{Q_{23} + k_2 V_2}
\]

\[
= \frac{W_2}{Q_{23} + k_2 V_2} + \frac{Q_{12} W_1}{(Q_{12} + k_1 V_1)(Q_{23} + k_2 V_2)}
\]
Non Steady State Solution

- **Impulse Load**
  - 1st lake
    \[ c_1 = c_{10} e^{-\lambda_4 t} \]
  - 2nd lake
    \[ c_2 = c_{20} e^{-\lambda_2 t} + c_{10} \frac{Q_{12}}{V_2 (\lambda_2 - \lambda_1)} \left( e^{-\lambda_4 t} - e^{-\lambda_2 t} \right) \]
  - 3rd lake
    \[
    c_3 = c_{30} e^{-\lambda_3 t} + c_{20} \frac{Q_{23}}{V_1 (\lambda_3 - \lambda_2)} \left( e^{-\lambda_4 t} - e^{-\lambda_3 t} \right) \\
    + c_{10} \frac{Q_{23} Q_{12}}{V_1 V_2 (\lambda_2 - \lambda_1)} \left( \frac{e^{-\lambda_4 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_1} - \frac{e^{-\lambda_2 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_2} \right)
    \]

Non Steady state solution

- **Step load**
  \[
  c_1 = \frac{W}{\dot{\lambda}_4 V_1} \left( 1 - e^{-\dot{\lambda}_4 t} \right) \\
  c_2 = \frac{Q_1 W}{V_2 \dot{\lambda}_1 V_1} \left[ \frac{1}{\dot{\lambda}_2} \left( 1 - e^{-\dot{\lambda}_2 t} \right) - \frac{1}{\dot{\lambda}_1 - \dot{\lambda}_2} \left( e^{-\dot{\lambda}_2 t} - e^{-\dot{\lambda}_1 t} \right) \right]
  \]
90Sr Fallout: Loading Function

- Close to an impulse load, centered around 1963
- Estimated value is: $7 \times 10^9 \text{ Ci/m}^2$
- Same for all lakes
Example: $^{90}$Sr fallout in Great Lakes

Non Steady State Solution

- Impulse Load
  - 1st lake
    \[ c_1 = c_{10} e^{-\lambda_1 t} \]
  - 2nd lake
    \[ c_2 = c_20 e^{-\lambda_2 t} + c_{10} \frac{Q_{12}}{V_2 (\lambda_2 - \lambda_1)} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \]
  - 3rd lake
    \[ c_3 = c_{30} e^{-\lambda_3 t} + c_{20} \frac{Q_{23}}{V_3 (\lambda_3 - \lambda_2)} (e^{-\lambda_2 t} - e^{-\lambda_3 t}) \]
    \[ + c_{10} \frac{Q_{23} Q_{12}}{V_3 V_2 (\lambda_2 - \lambda_1)} \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_1} - \frac{e^{-\lambda_2 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_2} \right) \]
Hydrologic Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Superior</th>
<th>Michigan</th>
<th>Huron</th>
<th>Erie</th>
<th>Ontario</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Depth</td>
<td>m</td>
<td>146</td>
<td>85</td>
<td>56</td>
<td>19</td>
<td>86</td>
</tr>
<tr>
<td>Surface Area</td>
<td>$10^6$ m$^2$</td>
<td>82,100</td>
<td>57,750</td>
<td>59,750</td>
<td>25,212</td>
<td>18,960</td>
</tr>
<tr>
<td>Volume</td>
<td>$10^9$ m$^3$</td>
<td>12,000</td>
<td>4,900</td>
<td>3,500</td>
<td>468</td>
<td>1,834</td>
</tr>
<tr>
<td>Outflow</td>
<td>$10^9$ m$^3$/yr</td>
<td>67</td>
<td>36</td>
<td>161</td>
<td>182</td>
<td>212</td>
</tr>
</tbody>
</table>

- Michigan $c_m = c_{11}^m$
- Superior $c_s = c_{11}^s$
- Huron $c_h = c_{21}^h + c_{22}^{sh} + c_{22}^{mh}$
- Erie $c_e = c_{31}^e + c_{32}^{he} + c_{33}^{hhe}$
- Ontario $c_o = c_{41}^o + c_{42}^{eo} + c_{43}^{hoe} + c_{44}^{theo} + c_{44}^{mheo}$

$^{90}$Sr Model

- From Chapra, 1997
Cambridge Example I

Cambridge Example II

- Summer showing stratification

E. Stony Brook Reservoir, August 5, 1998
Cambridge Example III

- Additional WQ Data

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Stratified Lake Model

\[
V_1 \frac{dc_1}{dt} = W_1 - Qc_1 + E'_{12}(c_2 - c_1) - k_1 V_1 c_1
\]

\[
V_2 \frac{dc_1}{dt} = W_2 + E'_{12}(c_1 - c_2) - k_2 V_2 c_2
\]
Analytical Solution

- Bulk vertical mixing coefficient

\[ E'_{12} = \frac{E_{12} A_{12}}{z_{12}} \]

- At steady state

\[ c_1 = \frac{(W_1 + \beta W_z)}{Q} \left( 1 + \frac{(1 - \beta) E'_{12}}{Q} \right) + \frac{V_1 k_1}{Q} \]

\[ c_2 = \beta \left( c_1 + \frac{W_2}{E'_{12}} \right) \]

Where: \( \beta = \frac{E'_{12}}{E'_{12} + V_2 k_2} \)

Estimation of Vertical Dispersion Coefficient

Use simple temperature model for Hypolimnion and analyze temperature data

\[ E_{12} = \frac{E_{12} A_{12} (T_1 - T_3)}{z_{12}} \]

\[ E_{12} = \frac{|T_2^{(i-1)} - T_2^{(i)}| V_2 z_{12}}{(T_1 - T_3) \Delta t A_{12}} \]
Seasonal Model for P

- Phosphorus separated into 2 components
  - SRP= soluble reactive phosphorus
  - NSRP= not soluble RP
    - this form can settle
- Lake is segmented into 2 layers
- Year can be divided into 2 seasons
- Mass balances are written for all 4 boxes
  - see Chapra, pg 554-555

Stratified Lake: 2 component model

Chapra, pg. 553
• To next lecture