Updated: 28 March 2013

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CEE 577: Surface Water Quality Modeling

Lecture #25

Lake Models & Lakes in Series

(Chapra, L5)

Lake Models

- Completely mixed analytical solutions (can accommodate time variable loading)
  - Single lake (lecture 4 & 5)
  - Lakes in Series
- Compartmentalized models
  - Two layer models (analytical)
  - 1, 2 and 3 dimensional approaches (numerical)

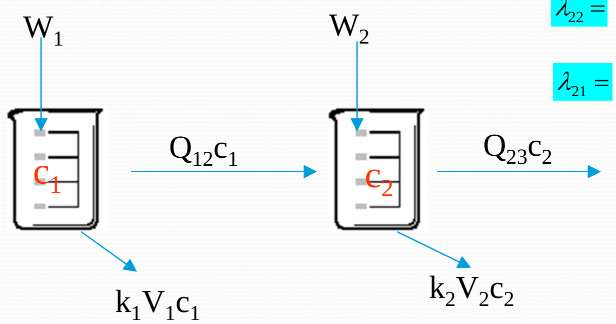
Chapra Reckhow

$\lambda_{11} = \lambda_1 = \frac{Q_{12}}{V_1} + k_1$

$\lambda_{22} = \lambda_2 = \frac{Q_{23}}{V_2} + k_2$

$\lambda_{21} = \frac{Q_{12}}{V_2}$

Completely mixed Lakes



$$V_1 \frac{dc_1}{dt} = W_1 - Q_{12}c_1 - k_1V_1c_1$$

$$V_2 \frac{dc_2}{dt} = W_2 + Q_{12}c_1 - Q_{23}c_2 - k_2V_2c_2$$

Steady state solution

$$W_1 = Q_{12}c_1 + k_1V_1c_1$$
$$W_2 = -Q_{12}c_1 + Q_{23}c_2 + k_2V_2c_2$$

$$c_1 = \frac{W_1}{Q_{12} + k_1V_1}$$

$$c_2 = \frac{W_2 + Q_{12}c_1}{Q_{23} + k_2V_2}$$

$$= \frac{W_2}{Q_{23} + k_2V_2} + \frac{Q_{12}W_1}{(Q_{12} + k_1V_1)(Q_{23} + k_2V_2)}$$

Non Steady State Solution

- Impulse Load
  - 1st lake $c_1 = c_{1o} e^{-\lambda_1 t}$
  - 2nd lake $c_2 = c_{2o} e^{-\lambda_2 t} + c_{1o} \dfrac{Q_{12}}{V_2(\lambda_2 - \lambda_1)} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$
  - 3rd lake

$$c_3 = c_{3o} e^{-\lambda_3 t} + c_{2o} \frac{Q_{23}}{V_3(\lambda_3 - \lambda_2)} \left(e^{-\lambda_2 t} - e^{-\lambda_3 t}\right) + c_{1o} \frac{Q_{23} Q_{12}}{V_3 V_2 (\lambda_2 - \lambda_1)} \left(\frac{e^{-\lambda_1 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_1} - \frac{e^{-\lambda_2 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_2} \right)$$

Non Steady state solution

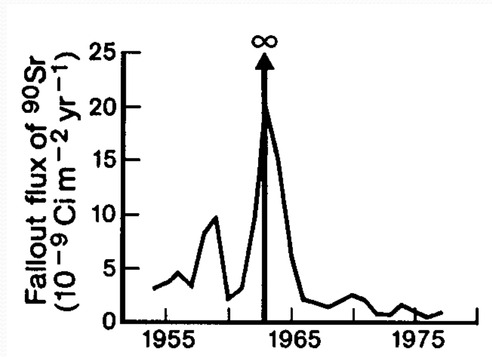
- Step load

$$c_1 = \frac{W}{\lambda_1 V_1} \left(1 - e^{-\lambda_1 t}\right)$$

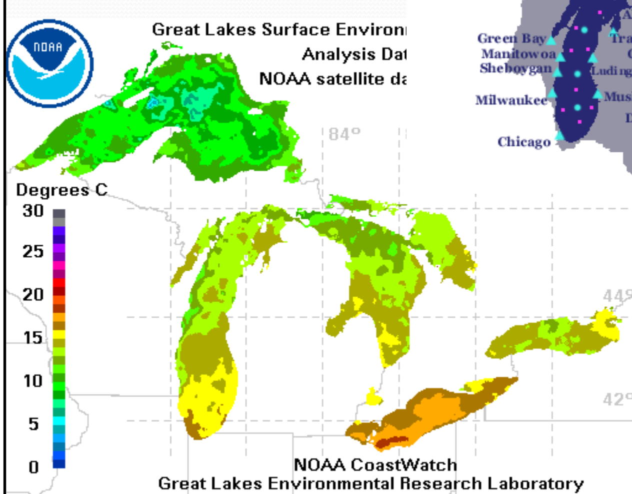
$$c_2 = \frac{Q_1}{V_2} \frac{W}{\lambda_1 V_1} \left[\frac{1}{\lambda_2} \left(1 - e^{-\lambda_2 t}\right) - \frac{1}{\lambda_1 - \lambda_2} \left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right) \right]$$

^{90}Sr Fallout: Loading Function

- Close to an impulse load, centered around 1963
- estimated value is: 70×10^{-9} Ci/m^2
- same for all lakes



Great Lakes



Example: ^{90}Sr fallout in Great Lakes

Non Steady State Solution

- Impulse Load
  - 1st lake
  - 2nd lake
  - 3rd lake

$$c_1 = \underbrace{c_{1o} e^{-\lambda_1 t}}_{C_{11}}$$

$$c_2 = \underbrace{c_{2o} e^{-\lambda_2 t}}_{C_{21}} + \underbrace{c_{1o} \frac{Q_{12}}{V_2(\lambda_2 - \lambda_1)} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right)}_{C_{22}}$$

$$c_3 = \underbrace{c_{3o} e^{-\lambda_3 t}}_{C_{31}} + \underbrace{c_{2o} \frac{Q_{23}}{V_3(\lambda_3 - \lambda_2)} \left(e^{-\lambda_2 t} - e^{-\lambda_3 t} \right)}_{C_{32}}$$
$$+ \underbrace{c_{1o} \frac{Q_{23} Q_{12}}{V_3 V_2 (\lambda_2 - \lambda_1)} \left(\frac{e^{-\lambda_1 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_1} - \frac{e^{-\lambda_2 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_2} \right)}_{C_{33}}$$

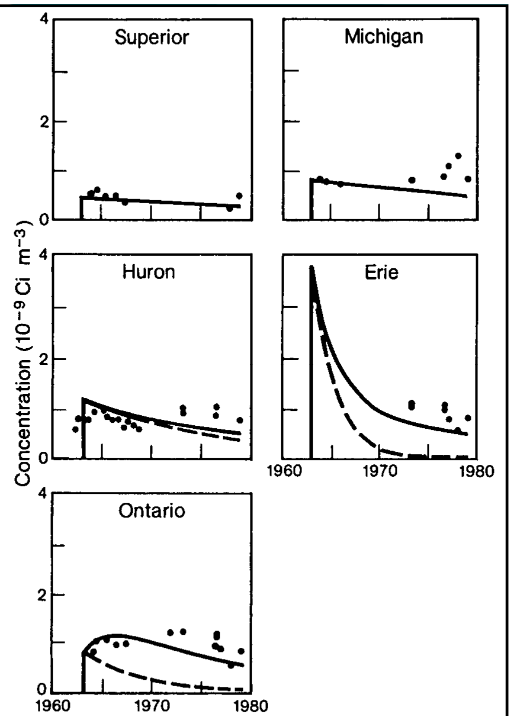
Hydrologic Parameters

Parameter	Units	Superior	Michigan	Huron	Erie	Ontario
Mean Depth	m	146	85	59	19	86
Surface Area	10^6 m^2	82,100	57,750	59,750	25,212	18,960
Volume	10^9 m^3	12,000	4,900	3,500	468	1,634
Outflow	10^9 m^3/yr	67	36	161	182	212

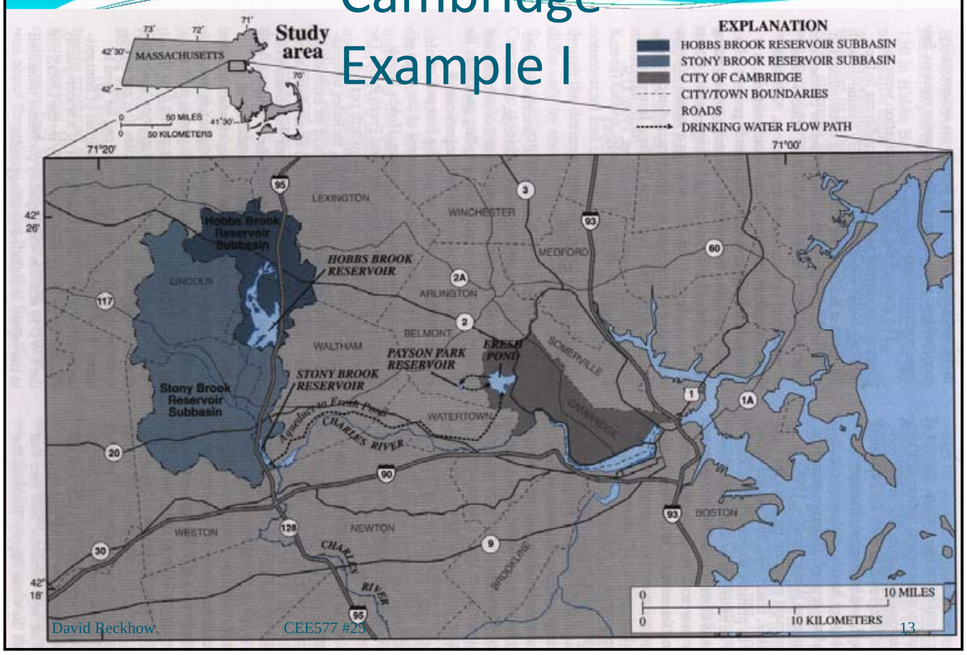
- Michigan $c_m = c_{11}^m$
- Superior $c_s = c_{11}^s$
- Huron $c_h = c_{21}^h + c_{22}^{sh} + c_{22}^{mh}$
- Erie $c_e = c_{31}^e + c_{32}^{he} + c_{33}^{she} + c_{33}^{mhe}$
- Ontario $c_o = c_{41}^o + c_{42}^{eo} + c_{43}^{heo} + c_{44}^{sheo} + c_{44}^{mheo}$

^{90}Sr Model

- From Chapra, 1997

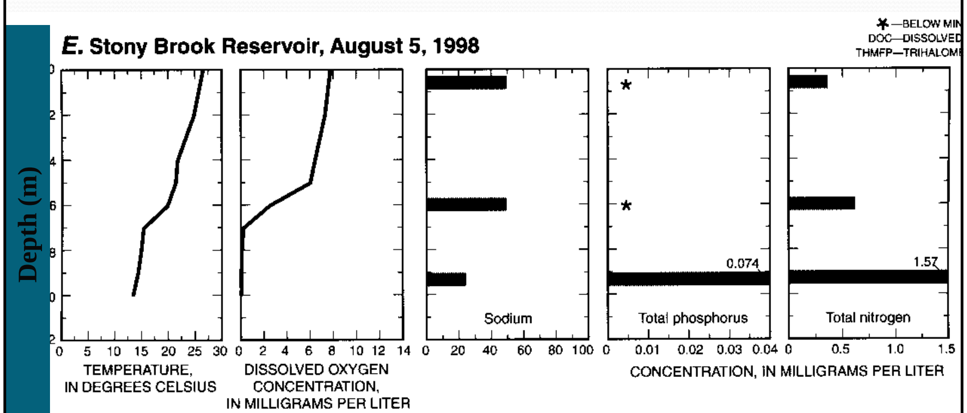


Cambridge Example I



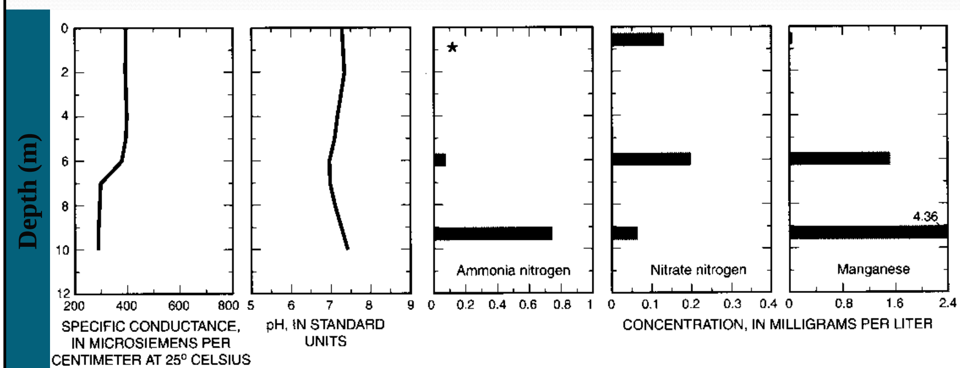
Cambridge Example II

- Summer showing stratification

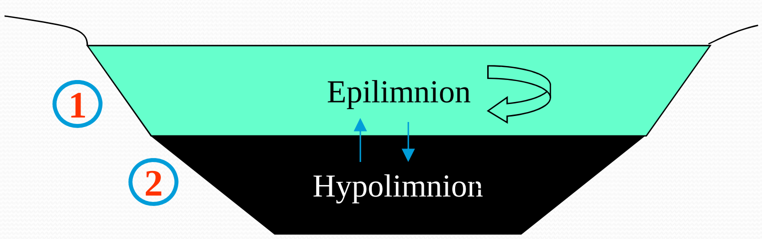


Cambridge Example III

- Additional WQ Data



Stratified Lake Model



$$V_1 \frac{dc_1}{dt} = W_1 - Qc_1 + E'_{12}(c_2 - c_1) - k_1 V_1 c_1$$

$$V_2 \frac{dc_1}{dt} = W_2 + E'_{12}(c_1 - c_2) - k_2 V_2 c_2$$

Analytical Solution

- Bulk vertical mixing coefficient
- At steady state

$$E'_{12} = \frac{E_{12} A_{12}}{z_{12}}$$

Vertical dispersion coefficient [L^2/T]

Interfacial Area [L^2]

Characteristic Length (~half the lake depth)

$$c_1 = \frac{(W_1 + \beta W_2)/Q}{\left(1 + (1-\beta)E'_{12}/Q\right) + V_1 k_1/Q}$$

$$c_2 = \beta\left(c_1 + \frac{W_2}{E'_{12}}\right)$$

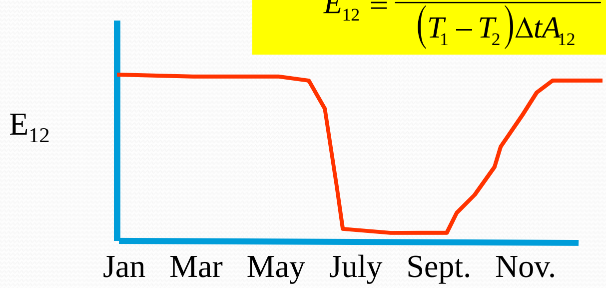
Where: $\beta = \dfrac{E'_{12}}{E'_{12} + V_2 k_2}$

Estimation of Vertical Dispersion Coefficient

Use simple temperature model for Hypolimnion and analyze temperature data

$$V_2\left(\frac{\Delta T_2}{\Delta t}\right) = \frac{E_{12} A_{12}}{z_{12}}\left(T_1 - T_2\right)$$

$$E_{12} = \frac{\left|T_2^{(t-1)} - T_2^{(t)}\right| V_2 z_{12}}{\left(T_1 - T_2\right)\Delta t A_{12}}$$

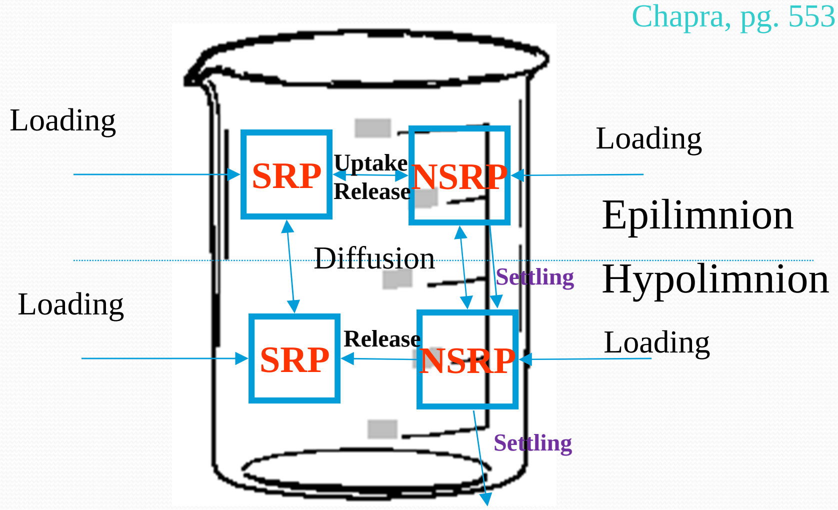


Seasonal Model for P

- Phosphorus separated into 2 components
  - SRP=soluble reactive phosphorus
  - NSRP=not soluble RP
    - this form can settle
- Lake is segmented into 2 layers
- Year can be divided into 2 seasons
- Mass balances are written for all 4 boxes
  - see Chapra, pg 554-555

Stratified Lake: 2 component model

Chapra, pg. 553



- To next lecture