Updated: 28 March 2013

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CEE 577: Surface Water Quality Modeling

Lecture #25

Lake Models & Lakes in Series

(Chapra, L5)

Lake Models

- Completely mixed analytical solutions (can accommodate time variable loading)
  - Single lake (lecture 4 & 5)
  - Lakes in Series
- Compartmentalized models
  - Two layer models (analytical)
  - 1, 2 and 3 dimensional approaches (numerical)

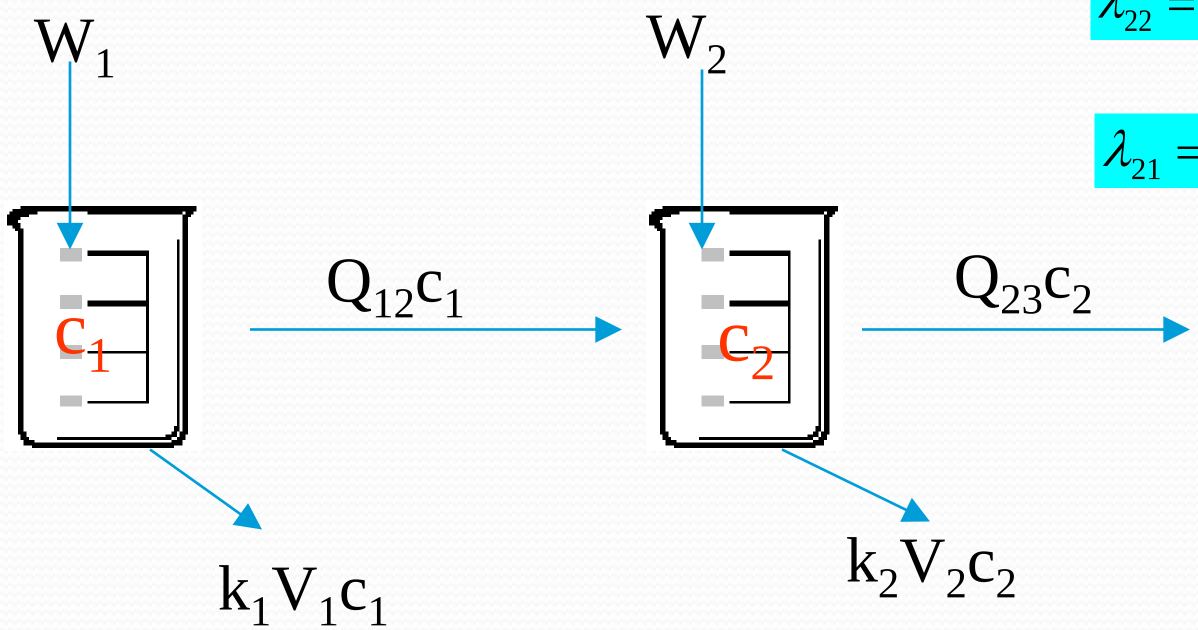
Chapra Reckhow

$$\boxed{\lambda_{11} =} \; \lambda_1 = \frac{Q_{12}}{V_1} + k_1$$

$$\boxed{\lambda_{22} =} \; \lambda_2 = \frac{Q_{23}}{V_2} + k_2$$

$$\boxed{\lambda_{21} =} \; \frac{Q_{12}}{V_2}$$

Completely mixed Lakes



$$V_1 \frac{dc_1}{dt} = W_1 - Q_{12}c_1 - k_1V_1c_1$$

$$V_2 \frac{dc_2}{dt} = W_2 + Q_{12}c_1 - Q_{23}c_2 - k_2V_2c_2$$

Steady state solution

$$W_1 = Q_{12}c_1 + k_1V_1c_1$$
$$W_2 = -Q_{12}c_1 + Q_{23}c_2 + k_2V_2c_2$$

$$c_1 = \frac{W_1}{Q_{12} + k_1V_1}$$
$$c_2 = \frac{W_2 + Q_{12}c_1}{Q_{23} + k_2V_2}$$
$$= \frac{W_2}{Q_{23} + k_2V_2} + \frac{Q_{12}W_1}{\left(Q_{12} + k_1V_1\right)\left(Q_{23} + k_2V_2\right)}$$

Non Steady State Solution

- Impulse Load
  - 1st lake
    $$c_1 = c_{1o} e^{-\lambda_1 t}$$
  - 2nd lake
    $$c_2 = c_{2o} e^{-\lambda_2 t} + c_{1o} \frac{Q_{12}}{V_2(\lambda_2 - \lambda_1)} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right)$$
  - 3rd lake

$$c_3 = c_{3o} e^{-\lambda_3 t} + c_{2o} \frac{Q_{23}}{V_3(\lambda_3 - \lambda_2)} \left(e^{-\lambda_2 t} - e^{-\lambda_3 t} \right) + c_{1o} \frac{Q_{23} Q_{12}}{V_3 V_2 (\lambda_2 - \lambda_1)} \left(\frac{e^{-\lambda_1 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_1} - \frac{e^{-\lambda_2 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_2} \right)$$

Non Steady state solution

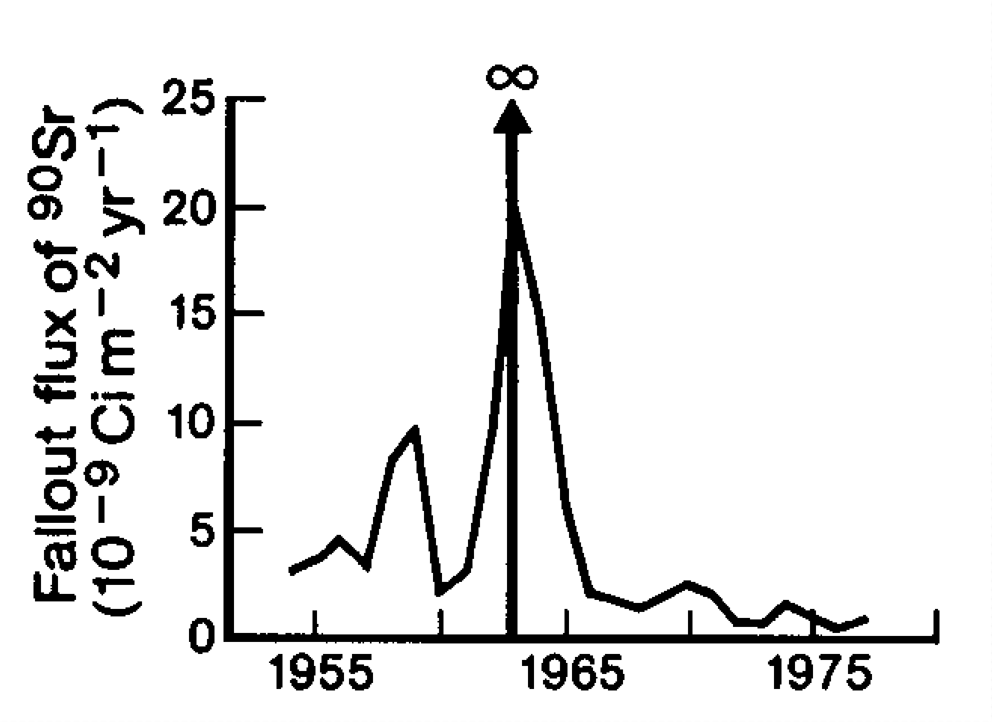
- Step load

$$c_1 = \frac{W}{\lambda_1 V_1}\left(1 - e^{-\lambda_1 t}\right)$$

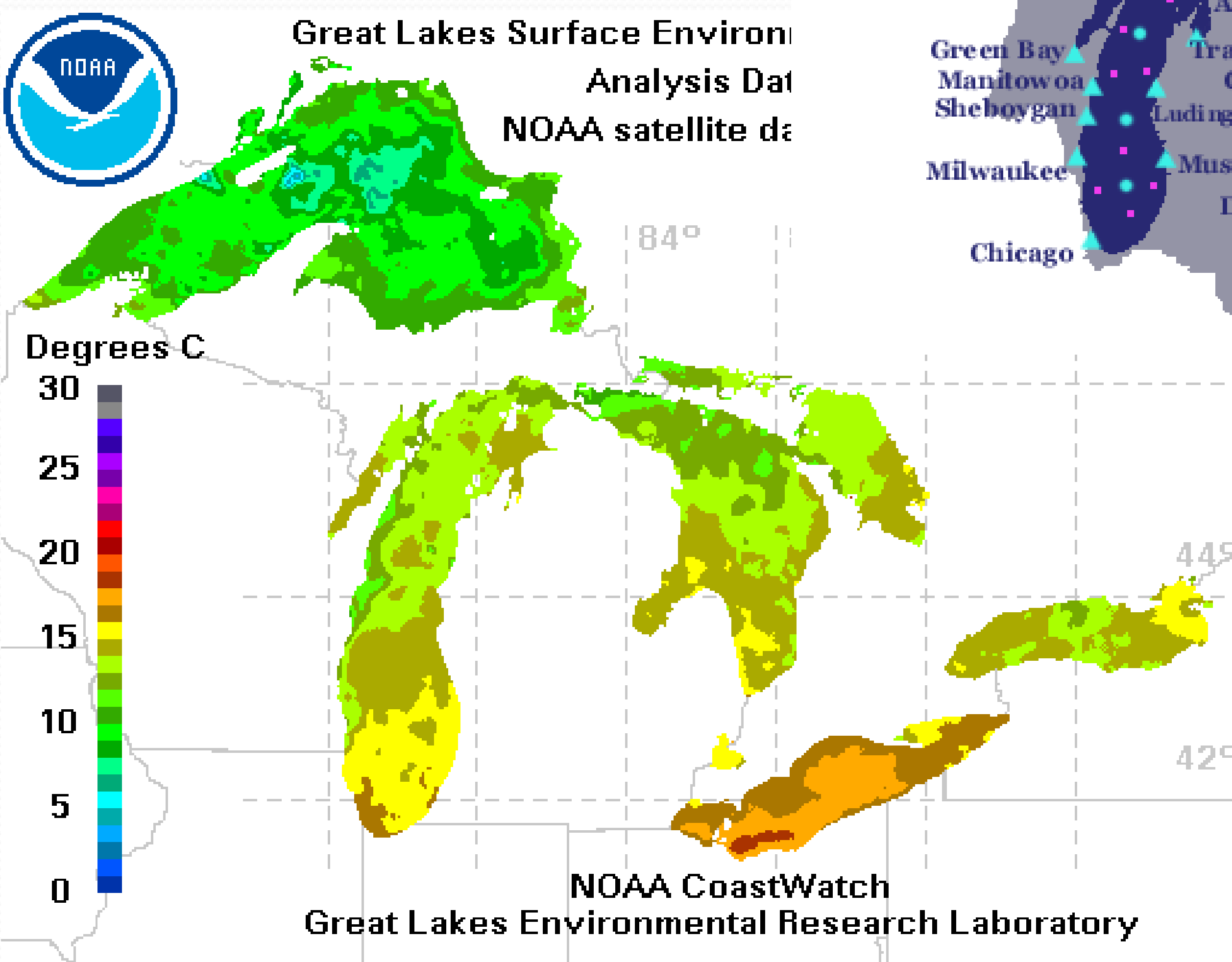
$$c_2 = \frac{Q_1}{V_2}\frac{W}{\lambda_1 V_1}\left[\frac{1}{\lambda_2}\left(1 - e^{-\lambda_2 t}\right) - \frac{1}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)\right]$$

^{90}Sr Fallout: Loading Function

- Close to an impulse load, centered around 1963
- estimated value is: $70x10^{-9}$ Ci/m²
- same for all lakes

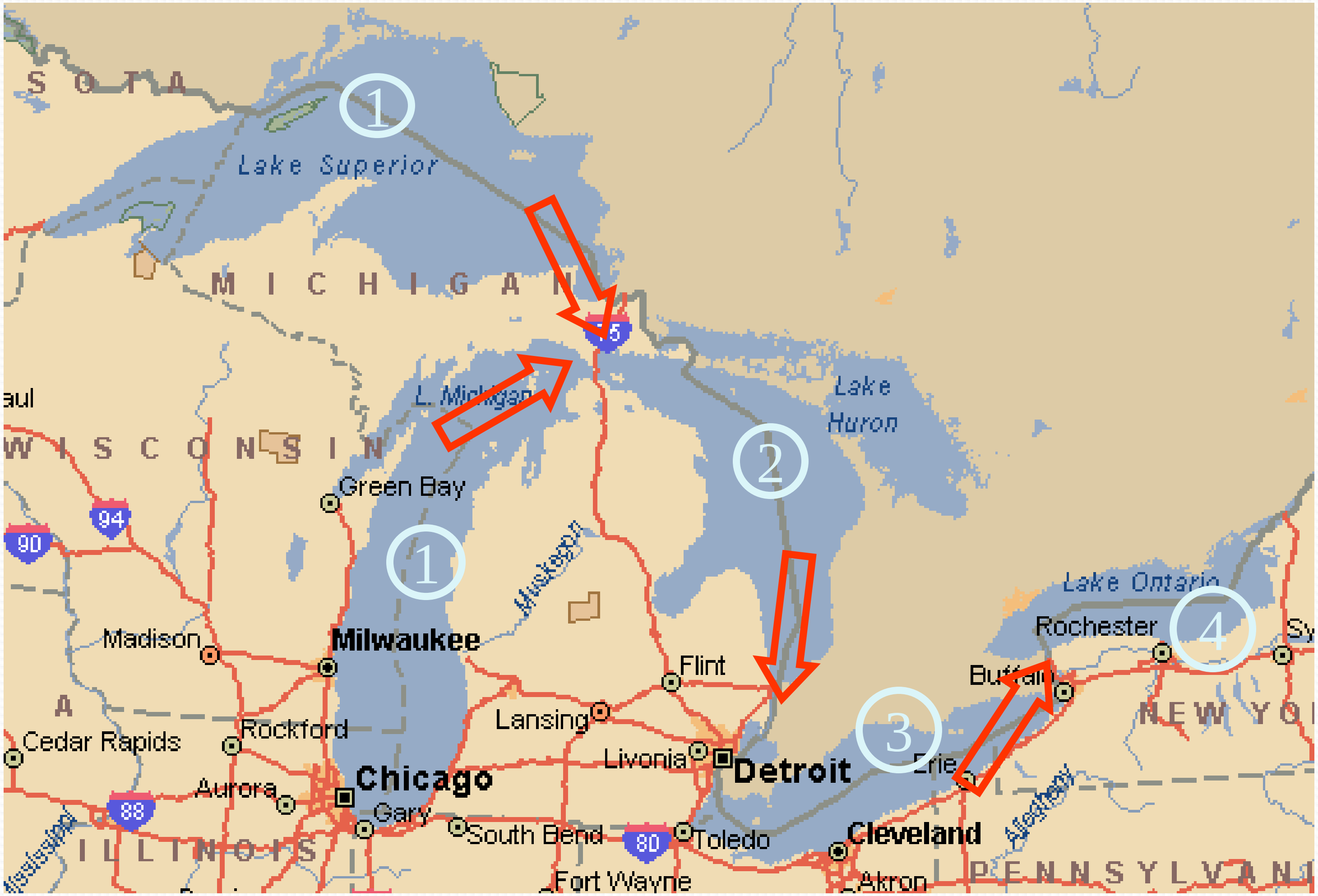


Great Lakes


Great Lakes Surface Environ
Analysis Dat
NOAA satellite da
NOAA
Degrees C
30
25
20
15
10
5
0
84°
44°
42°
NOAA CoastWatch
Great Lakes Environmental Research Laboratory


Great Lakes Surveillance Stations
Master Stations
Surveillance Stations
Grand Marais
Duluth
Ontonagon
Marquette
Alpena
Green Bay
Traverse City
Manitowoa
Sheboygan
Ludington
Bay City
Port Huron
Milwaukee
Muskegon
Buffalo
Detroit
Chicago
Erie
Toledo
Cleveland

Example: ^{90}Sr fallout in Great Lakes



Non Steady State Solution

- Impulse Load
  - 1st lake
  - 2nd lake
  - 3rd lake

$$c_1 = \underbrace{c_{1o} e^{-\lambda_1 t}}_{C_{11}}$$

$$c_2 = \underbrace{c_{2o} e^{-\lambda_2 t}}_{C_{21}} + \underbrace{\boxed{c_{1o} \frac{Q_{12}}{V_2(\lambda_2 - \lambda_1)} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)}}_{C_{22}}$$

$$c_3 = \underbrace{c_{3o} e^{-\lambda_3 t}}_{C_{31}} + \underbrace{\boxed{c_{2o} \frac{Q_{23}}{V_3(\lambda_3 - \lambda_2)} \left(e^{-\lambda_2 t} - e^{-\lambda_3 t}\right)}}_{C_{32}}$$

$$+ \underbrace{\boxed{c_{1o} \frac{Q_{23} Q_{12}}{V_3 V_2 (\lambda_2 - \lambda_1)} \left(\frac{e^{-\lambda_1 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_1} - \frac{e^{-\lambda_2 t} - e^{-\lambda_3 t}}{\lambda_3 - \lambda_2} \right)}}_{C_{33}}$$

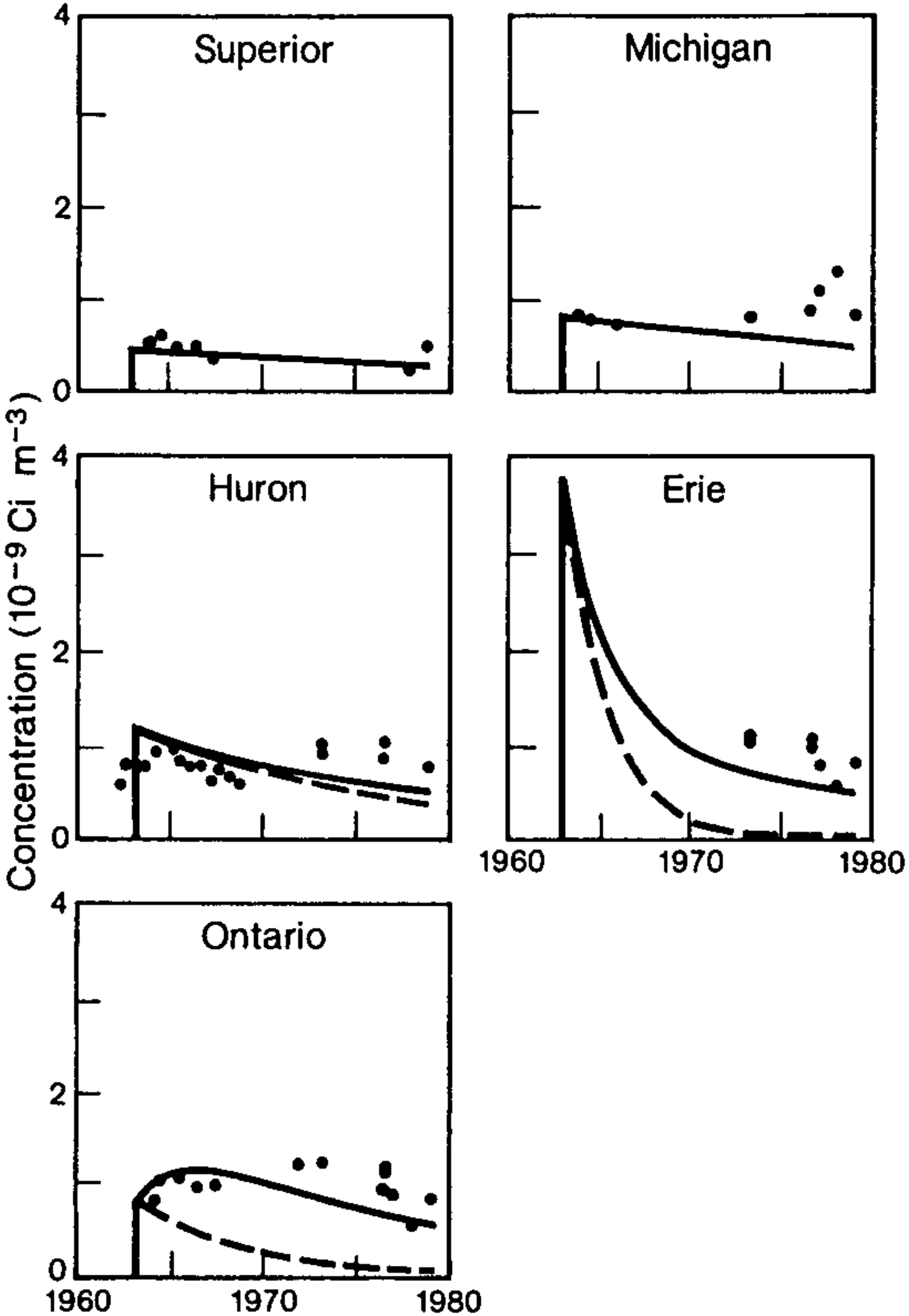
Hydrologic Parameters

Parameter	Units	Superior	Michigan	Huron	Erie	Ontario
Mean Depth	m	146	85	59	19	86
Surface Area	10^6 m^2	82,100	57,750	59,750	25,212	18,960
Volume	10^9 m^3	12,000	4,900	3,500	468	1,634
Outflow	10^9 m^3/yr	67	36	161	182	212

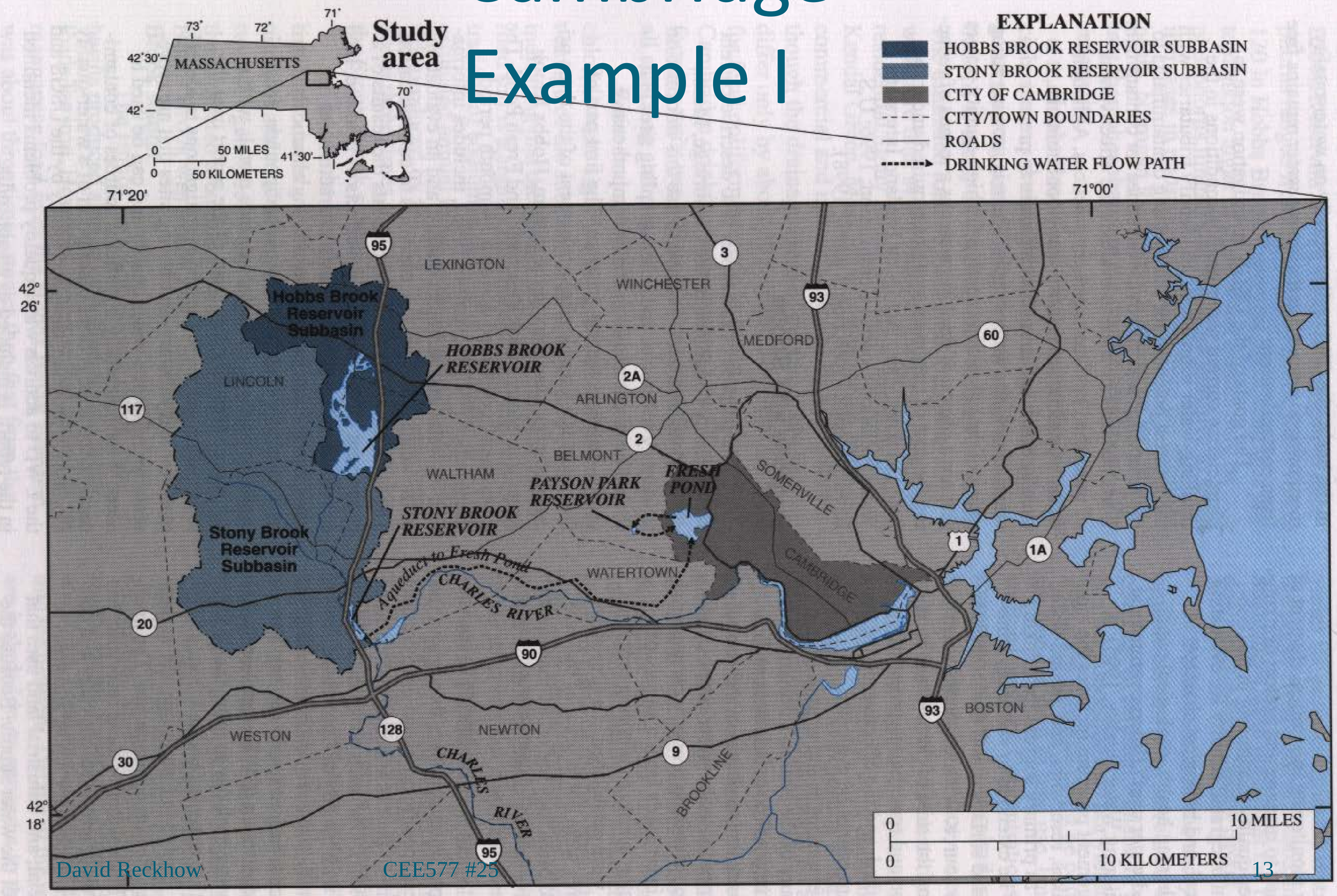
- Michigan $c_m = c_{11}^m$
- Superior $c_s = c_{11}^s$
- Huron $c_h = c_{21}^h + c_{22}^{sh} + c_{22}^{mh}$
- Erie $c_e = c_{31}^e + c_{32}^{he} + c_{33}^{she} + c_{33}^{mhe}$
- Ontario $c_o = c_{41}^o + c_{42}^{eo} + c_{43}^{heo} + c_{44}^{sheo} + c_{44}^{mheo}$

^{90}Sr Model

- From Chapra, 1997

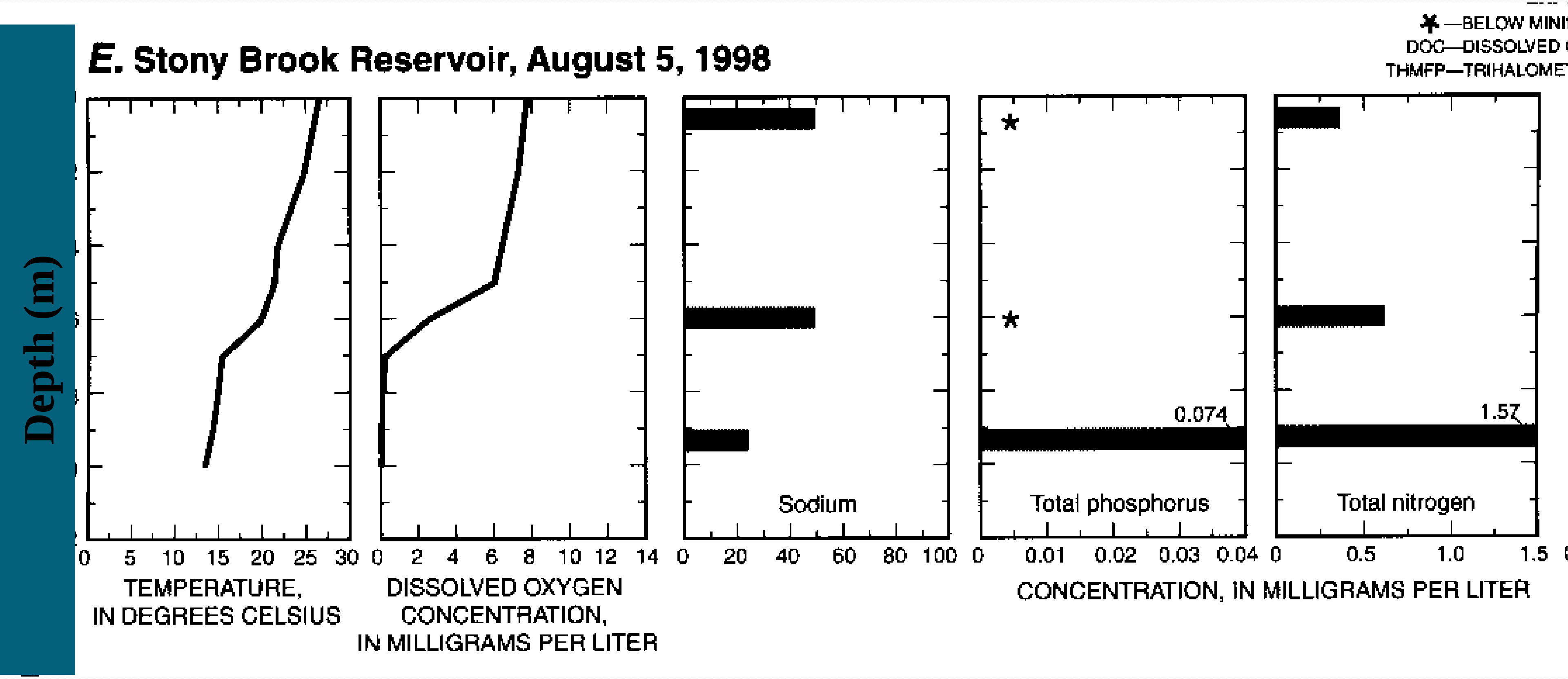


Cambridge Example I



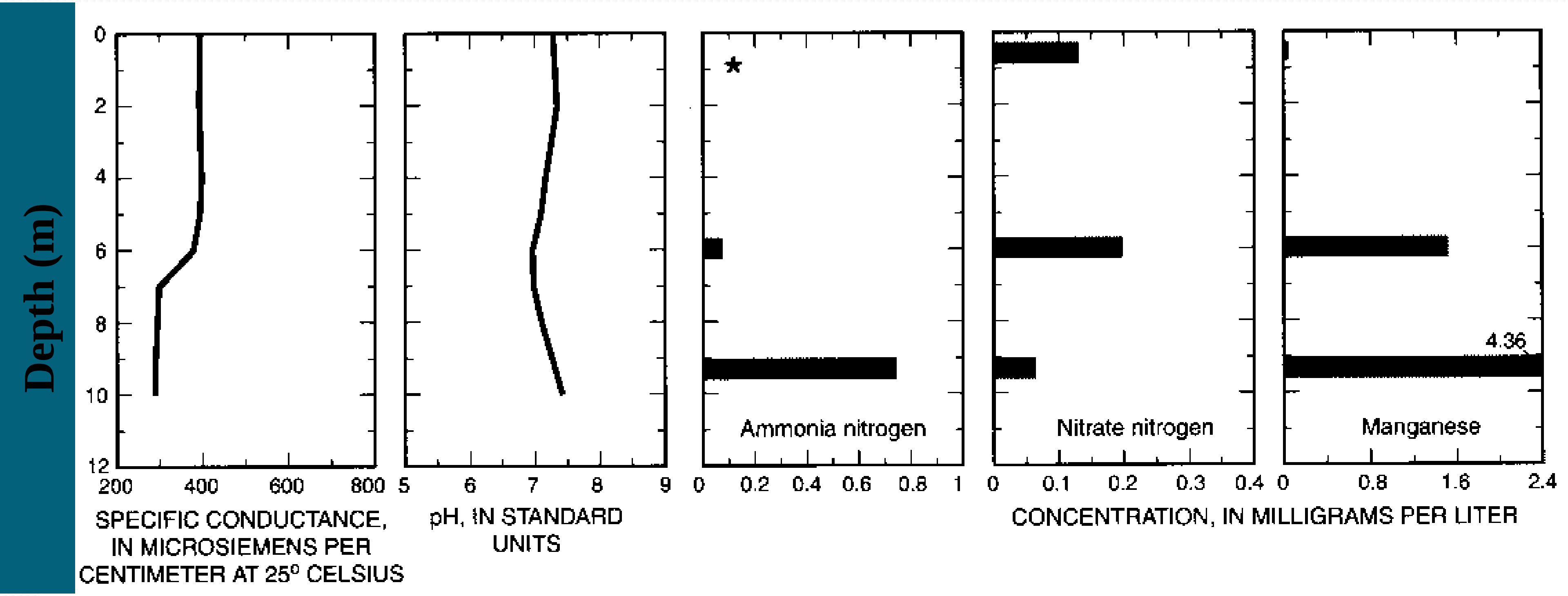
Cambridge Example II

- Summer showing stratification

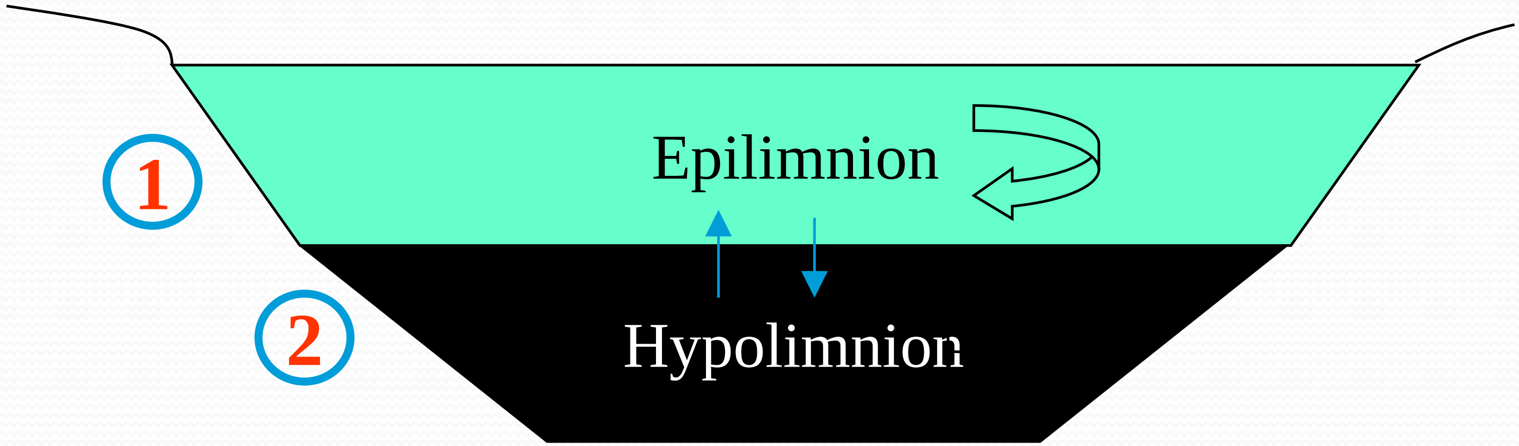


Cambridge Example III

- Additional WQ Data



Stratified Lake Model



$$V_1 \frac{dc_1}{dt} = W_1 - Qc_1 + E'_{12}(c_2 - c_1) - k_1 V_1 c_1$$

$$V_2 \frac{dc_1}{dt} = W_2 + E'_{12}(c_1 - c_2) - k_2 V_2 c_2$$

Analytical Solution

- Bulk vertical mixing coefficient

- At steady state

$$E'_{12} = \frac{E_{12} A_{12}}{z_{12}}$$

Vertical dispersion coefficient [L^2/T]

Interfacial Area [L^2]

Characteristic Length (~half the lake depth)

$$c_1 = \frac{(W_1 + \beta W_2)/Q}{\left(1 + (1-\beta)\dfrac{E'_{12}}{Q}\right) + \dfrac{V_1 k_1}{Q}}$$

$$c_2 = \beta\left(c_1 + \frac{W_2}{E'_{12}}\right)$$

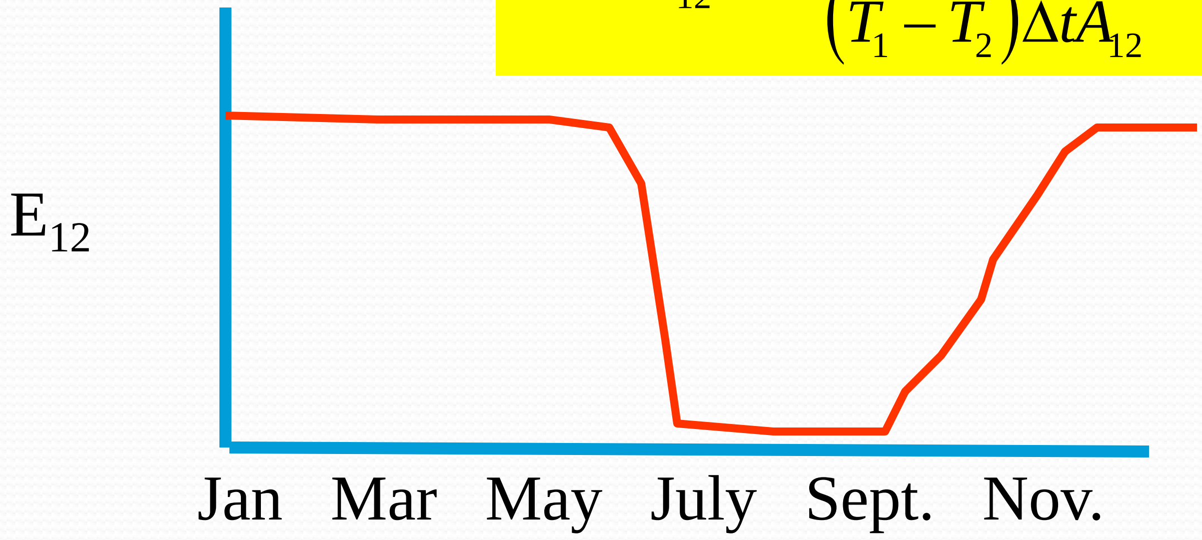
Where: $\beta = \dfrac{E'_{12}}{E'_{12} + V_2 k_2}$

Estimation of Vertical Dispersion Coefficient

Use simple temperature model for Hypolimnion and analyze temperature data

$$V_2\left(\frac{\Delta T_2}{\Delta t}\right)=\frac{E_{12}A_{12}}{z_{12}}\left(T_1-T_2\right)$$

$$E_{12}=\frac{\left|T_2^{(t-1)}-T_2^{(t)}\right|V_2 z_{12}}{\left(T_1-T_2\right)\Delta t A_{12}}$$

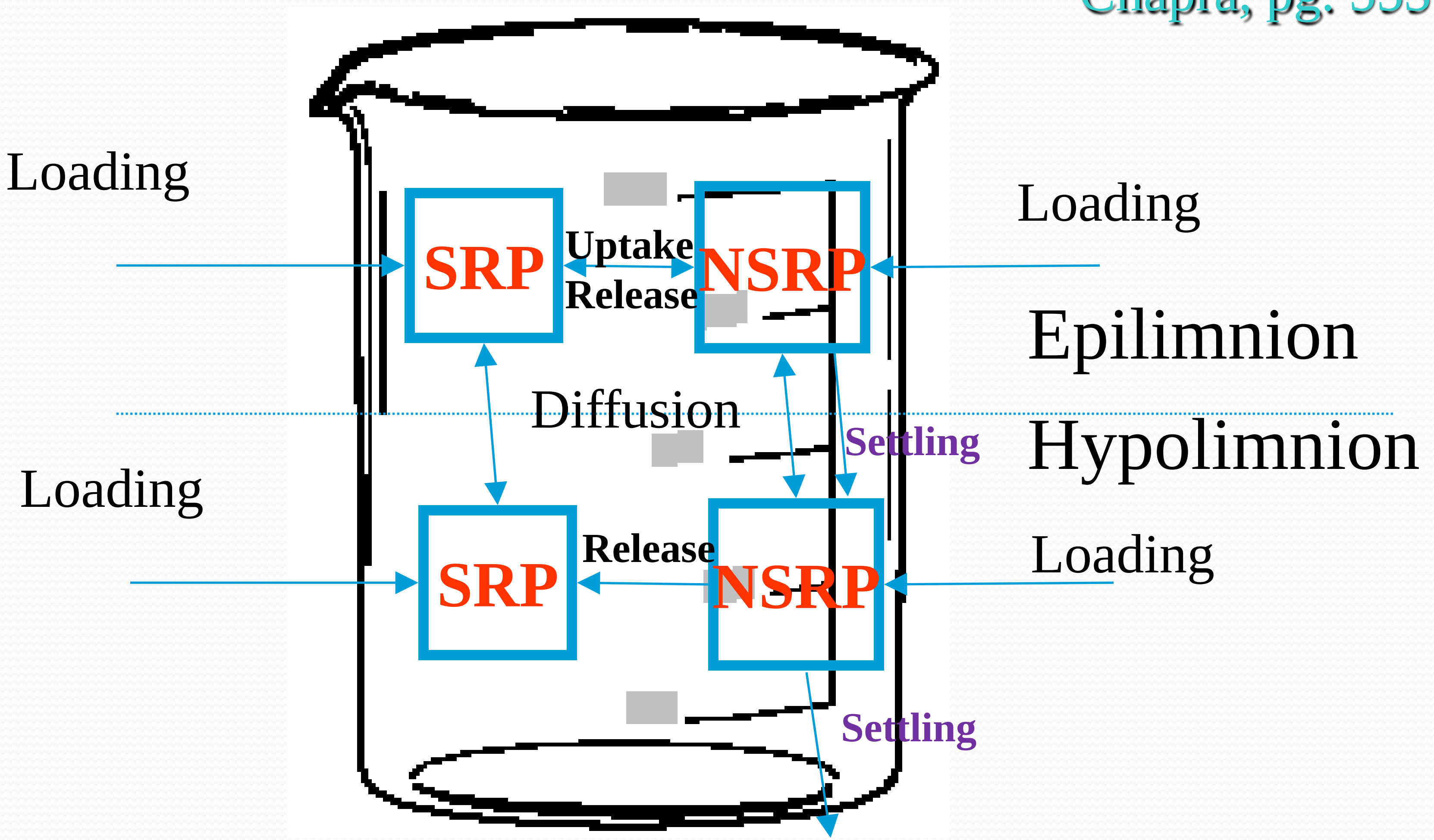


Seasonal Model for P

- Phosphorus separated into 2 components
  - SRP=soluble reactive phosphorus
  - NSRP=not soluble RP
    - this form can settle
- Lake is segmented into 2 layers
- Year can be divided into 2 seasons
- Mass balances are written for all 4 boxes
  - see Chapra, pg 554-555

Stratified Lake: 2 component model

Chapra, pg. 553



- To next lecture