Updated: 27 January 2013

Print version

CEE 577: Surface Water Quality Modeling

Lecture #5

(particular solutions)

Chapra L4

Non-steady state solutions

- After a change in loading
  - how long will it take for WQ to improve?
  - what will the shape of the curve look like?
- Return to the mass balance and solve

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_s c$$

Divide by V

$$\frac{dc}{dt} = \frac{W(t)}{V} - \frac{Q}{V}c - kc - \frac{v}{H}c$$

An eigenvalue

"the cleansing rate"

$$\frac{dc}{dt} + \lambda c = \frac{W(t)}{V}$$

Where: $\lambda = \frac{Q}{V} + k + \frac{v}{H}$

$\lambda \equiv \frac{\alpha}{V}$

Approximate Solution

- Non-steady state lake is described by an ODE:

$$\underbrace{\frac{dc}{dt} = \frac{W(t)}{V} - \frac{Q}{V}c - kc - \frac{v}{H}c}_{c_{t+\Delta t} = c_t + \frac{dc}{dt}\Delta t}$$

  - assuming you know the initial concentrations and coefficients, you can project future concentration numerically
  - some error because dc/dt changes continuously with c

Exact Solution to MB equation

$c = c_g + c_p$ The general and particular solutions

- The general solution
  - where W(t)=o

$$\boxed{c \equiv c_g}$$

$$\frac{dc}{dt} + \lambda c = 0$$

$$\frac{dc}{c} = -\lambda dt$$

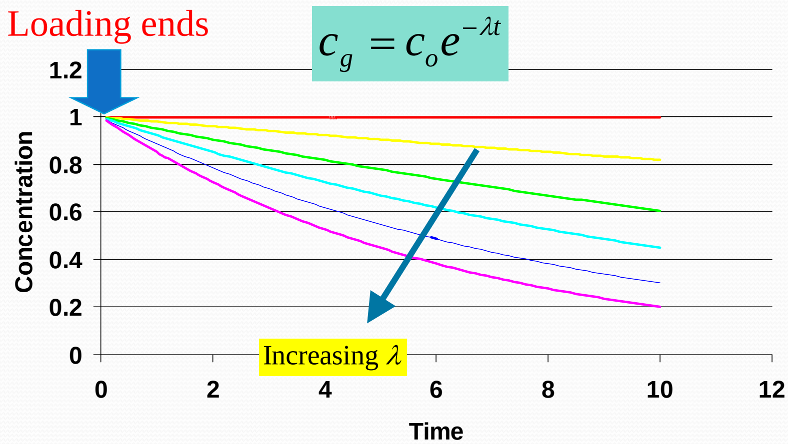
$$\int_{c_o}^{c} \frac{dc}{c} = -\lambda \int_0^t dt$$

$$\ln c\Big|_{c_o}^{c} = -\lambda t\Big|_0^t$$

$$\ln c = \ln c_o - \lambda t$$

$$c_g \equiv c = c_o e^{-\lambda t}$$

General Solution



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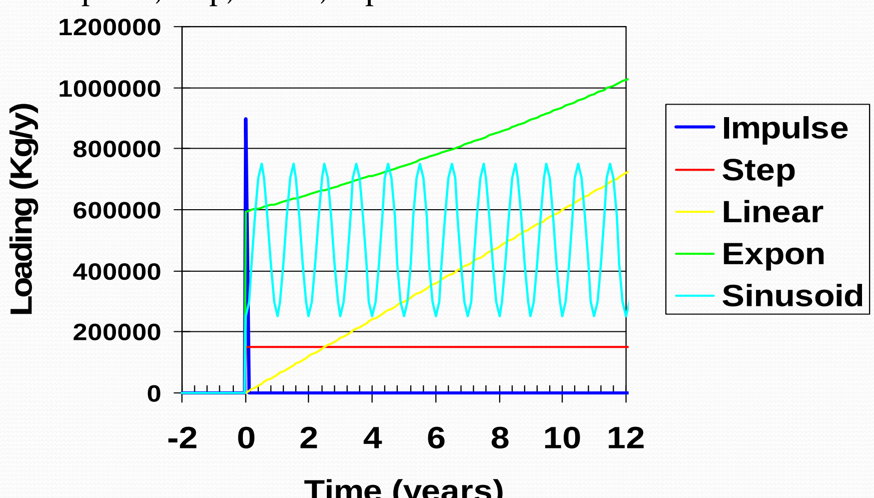
Loading Types

- Impulse, step, linear, exponential & sinusoidal



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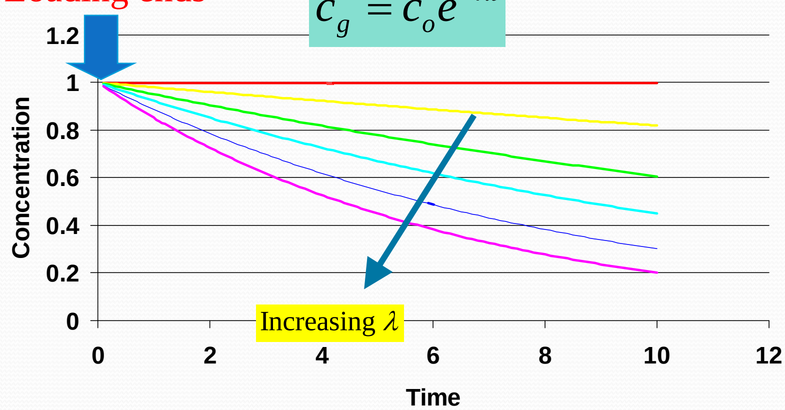
General Solution

$$c_g = c_o e^{-\lambda t}$$

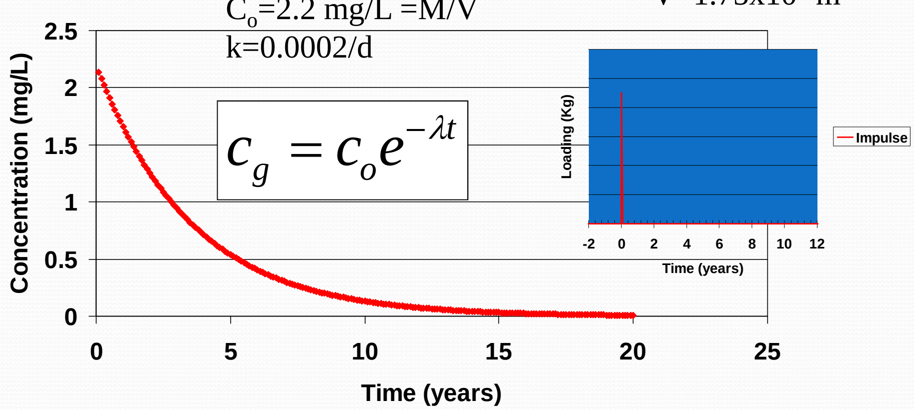


Impulse Loading

M=3.85x10^6 Kg
Q=10^6 m^3/d
A=1.1x10^8 m^2
V=1.75x10^9 m^3

C_o=2.2 mg/L =M/V
k=0.0002/d

$$c_g = c_o e^{-\lambda t}$$



Solving for the particular solution

- Use the integrating factor method

$$\frac{dx}{dy} + p(y)x = g(y)$$

$$I = e^{\int p(y)dy}$$

$$\frac{dc}{dt} + \lambda c = \frac{W(t)}{V}$$

$$I = e^{\int \lambda dt} = e^{\lambda t}$$

$$e^{\lambda t}\frac{dc}{dt} + e^{\lambda t}\lambda c = e^{\lambda t}\frac{W(t)}{V}$$

$$\frac{d}{dt}\left(e^{\lambda t}c\right) = e^{\lambda t}\frac{dc}{dt} + e^{\lambda t}\lambda c$$

$$\frac{d}{dt}\left(e^{\lambda t}c\right) = e^{\lambda t}\frac{W(t)}{V}$$

$$e^{\lambda t}c\Big|_o^t = \frac{1}{V}\int e^{\lambda t}W(t)dt$$

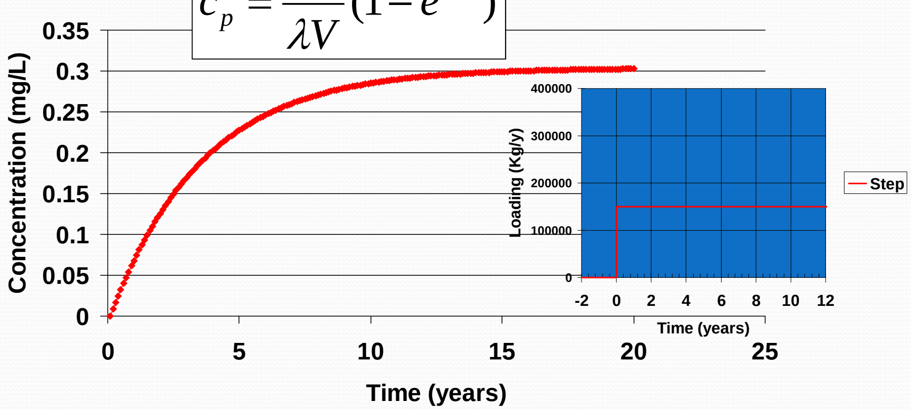
$$ce^{\lambda t} - c_o = \frac{1}{V}\int e^{\lambda t}W(t)dt$$

$$c(t) = \underbrace{\boxed{c_o e^{-\lambda t}}}_{c_g} + \underbrace{\boxed{\frac{e^{-\lambda t}}{V}\int_o^t e^{\lambda t}W(t)dt}}_{c_p}$$

Step Loading

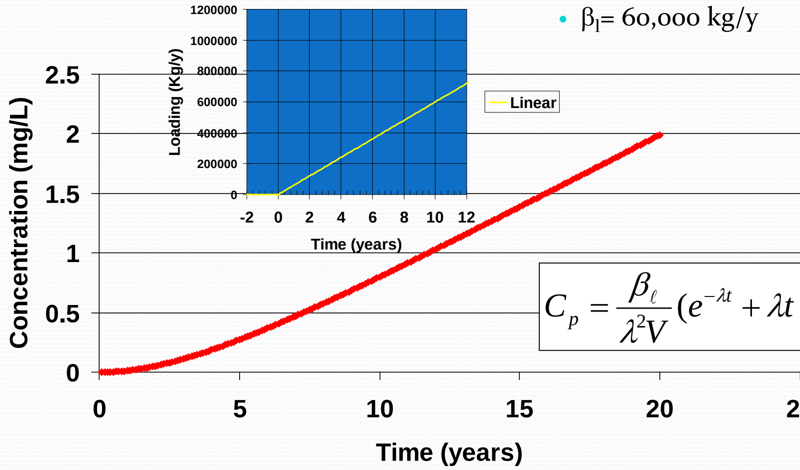
- W=410 kg/d

$$c_p = \frac{\overline{W}}{\lambda V}(1 - e^{-\lambda t})$$



Linear Loading

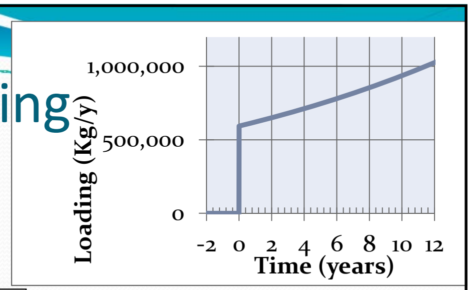
- **$W(t)=\beta_l t$**
  - β_l= 60,000 kg/y



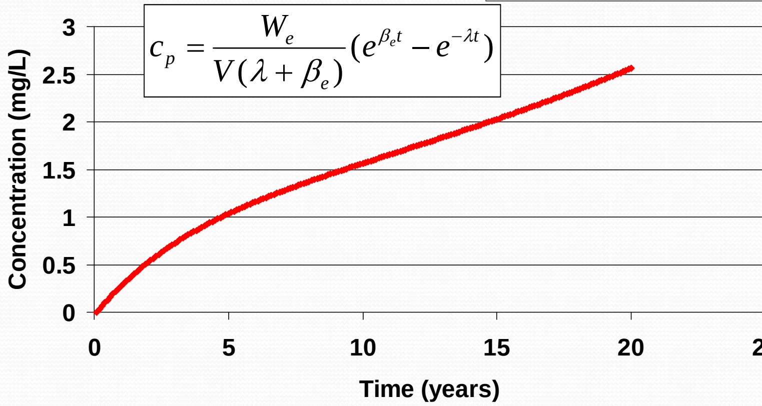
$$C_p = \frac{\beta_\ell}{\lambda^2 V}(e^{-\lambda t} + \lambda t - 1)$$

Exponential Loading

- **$W(t)=W_e e^{\beta_e t}$**
  - W_e= 1625 kg/d
  - β_e= 0.04558 /yr



$$c_p = \frac{W_e}{V(\lambda + \beta_e)}(e^{\beta_e t} - e^{-\lambda t})$$



- To next lecture

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