

# **Analysis of LWR model with fundamental diagram subject to uncertainties**

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**Abstract**

The LWR model is of interest since it can successfully reproduce some essential features of traffic flow, such as the formation and propagation of various waves. Recently, however, the disadvantageous aspects of traditional LWR model are attracting more and more attention. Many researchers have made great effort to develop new models based on the LWR model, such that some prominent nonlinear characteristics of traffic flow, such as platoon diffusion, capacity drop, hysteresis, and spontaneous onset of congestion are captured.

In this paper, we investigate the LWR model from an uncertainty perspective. Rather than developing a new explanatory model, we attempt to analyze how reliable the LWR model prediction will be if the fundamental diagram (FD) used is not specified accurately. Our work reflects the current debate on the nature of the FD. To be specific, we postulate a form of the flux function driven by random free flow speed, which accommodates the scattering feature observed in the speed-density plot. We show the properties of the LWR model with the new flux. A third-order essentially non-oscillatory (ENO) finite difference algorithm is devised to solve the model. An approach to evaluate the predictability of traffic disturbance propagation with this model is presented. We interpret the results and conclude that if the FD cannot be undoubtedly specified, the LWR model will deteriorate and only make a reasonable prediction in relatively short time scale.

# 1 Introduction

## 1.1 Background

The LWR model[1][2], as the simplest kinetic wave (KW) model, is central in many investigations associated with traffic dynamics and control. This is probably because some essential features of traffic flow, such as wave formation and propagation, can be qualitatively well reproduced with the LWR model. The reader is referred to [3][4], and the references therein for full discussion of this model and its variations.

At the same time, the deficiencies of the LWR model are well known. For example, the LWR model fails to generate capacity drop, hysteresis, relaxation, platoon diffusion, or spontaneous congestion. The existence of these traits have been confirmed in many field observations. In the LWR model, mass (i.e., the vehicles) conservation always holds true provided that no sinks and sources are involved. Thus to remedy the problem, one may either follow the rationale of hydrodynamics and add ‘momentum conservation’ type equations, or question the initial assumptions regarding the fundamental diagram (FD), i.e., the smoothness and concavity, and propose other forms of models. These two strategies are, in spirit, consistent with the discussions in [4], where they are named ‘higher-order’ and ‘lower-order’ extensions of LWR model, respectively.

Of the two strategies, the former one seems much more debateable. Daganzo[3] describes the logical flaws in the argument of deriving high-order continuum models, and he shows that negative flow and speed could be unreasonably generated (this means anisotropy property will be violated). Inconsistency inherent in such model in a statistical perspective is also demonstrated in [5]. Zhang[4] holds that the two strategies are very close in nature, regarding the anisotropy issue as well as necessity to introduce behavioral laws, and he finds that the models investigated therein can all be reduced to a KW model endowed with an appropriate ‘effective fundamental diagram’. Moreover, the plots of speed-concentration or flow-concentration usually exhibit structured randomness/scatterings, especially around congestion region. These observations, though not necessarily conflicting the initial assumption of the FD, are quite common and make thorough analysis possible. Therefore, it seems to be at least a good starting point to follow the latter strategy and see what may happen.

Different interpretations of the scattering have lead to many new models, either explaining this effect itself or incorporating this effect to obtain traffic flow models with new features. On one hand, Castillo and Benítez[6] and Cassidy[7] demonstrate that well-defined and reproducible relations of traffic variables, such as flow-occupancy, can be obtained only if the stationary traffic data are used. This implies the scattering is due to the existence of transient states. Zhang[4] shows, at the price of lowering data resolution, a well-behaved FD can also be constructed from locally smoothed transient traffic data. On the other hand, the hysteresis phenomenon has been well known[8][9]. This indicates speed-concentration curve should consist of hysteresis loops rather than a single curve. Meanwhile, it is noted that Newell’s extension of the FD[10] and the reversed- $\lambda$  shaped FD have been in existence for a long time. Nonetheless, the FD with structured variability still seems to be a puzzling issue in recent years and continues triggering discussions in different dimensions. Treiber and Helbing[11] show that conventional measurement methods with the delay of driving behavior may result in the scattering of flow-density data in ‘synchronized’ congested traffic. Zhang and Kim[12][13] utilize car-following models and explain the occurrence of capacity drop and hysteresis with one variable of gap time. Wong and Wong[14] and Ngoduy

1  
2 and Liu[15] attribute nonlinear traffic phenomena such as hysteresis, capacity drop, and dispersion  
3 of traffic platoon to the distribution of heterogeneous drivers and they formulate this idea into multi-  
4 class continuum traffic models. Kerner et al.[16][17][18] criticize the FD approach, arguing that  
5 this approach fails to produce the spatial-temporal features of congestion correctly. The concept of  
6 ‘synchronized flow’ is introduced and the three-phase traffic flow theory is developed. This list of  
7 works is not intended to be complete in any sense. One fact is evident that the concept of the FD,  
8 though seemingly basic, is hardly unequivocal since its birth. Many interesting findings regarding  
9 traffic flow are initiated with the fresh insight into the FD.

10 The aim of this paper, rather than resolve those existing controversies with FDs, is to inves-  
11 tigate the potential influences brought by such ambiguity. To fulfill this end, we analyze the pos-  
12 sible sources of randomness with the FD first, and then following the methodology of uncertainty  
13 analysis, develop a numerical procedure to evaluate how the randomness affects the faithfulness  
14 of the LWR model. Though not attempting to be explanatory, our study essentially unveils the  
15 robustness of LWR model to the imperfect specification of the FDs. Knowledge of this hopefully  
16 prompts further understanding of traffic flow features.

## 17 18 19 **1.2 Paper organization**

20 The remainder of this paper is organized as follows. In Section 2, we first present a brief account for  
21 uncertainty analysis and differentiate the source of randomness associated with the speed-density  
22 plot (Subsection 2.1). In this context, we discuss the extension of the FD from a single curve to  
23 random function, which accommodates a distribution of flow for any given density value. Then  
24 we propose a specific extension that assumes the specification of free flow is subject to uncertainty  
25 (Subsection 2.2). Reasons and implication of introducing such assumption are detailed therein,  
26 in a perspective of interpolation. Based on the proposed random FD, we conduct a numerical  
27 study (Section 3). The model to be solved is the LWR with a random FD, and an essentially  
28 non-oscillatory (ENO) finite difference scheme is devised to fulfill the purpose (Subsection 3.1).  
29 Subsequently, we design an example with local jam/vacuum initial data, and attempt to find how  
30 predictable are the propagations of these local disturbances with the previously discussed uncertain  
31 setting (Subsection 3.2). In Section 4, we briefly summarize the paper and show the gaps we find.

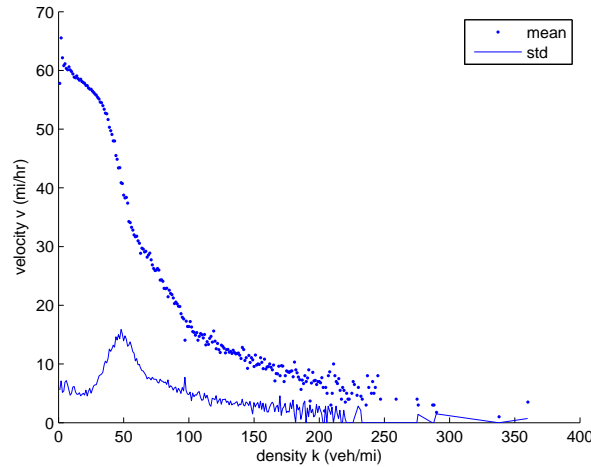
## 32 33 34 **2 The LWR model in stochastic setting**

### 35 36 37 **2.1 Uncertainty analysis**

38 In system modeling literature, the uncertainty is commonly defined as *any deviation from the un-*  
39 *achievable ideal of completely deterministic knowledge of the relevant system*[19]. Roughly speak-  
40 ing, the uncertainty analysis is to model the uncertainty with a system and understand the related  
41 influences. Risk and reliability assessment are among the most significant examples of uncertainty  
42 analysis, and find their applications in hydrology, structure engineering and economics, etc. The  
43 uncertainty analysis in the general context of conservation laws has been reported in literature for  
44 two decades. Many of them are in the spirit of randomizing the flux function, either spatially or  
45 temporally. For example, in [20][21], the authors discuss the expectation and convergence of a  
46 stochastic Buckley-Leverett equation. In [22] the large-time property of Burger’s equation is ana-  
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lyzed. More recently, Bale et al. presented a general solution approach, without directly addressing the issue of randomness, for the conservation equation with flux function admitting spatial variation [23]. All of them have focused on the spatial variability of the flux function. In the traffic field, Jou and Lo considered a nonlinear macroscopic traffic flow equation perturbed by a Brownian motion [24].

Before unfolding our investigation, empirical evidence is presented. For any empirical speed-density relation, the mapping from density to speed is always multi-valued after rounding off the density data. This enables us to calculate the mean and standard deviation of the speed indexed by density. In Figure 1 the mean and standard deviation of the speed versus density at one station from GA-400 ITS dataset (which are collected by virtual loop detectors on freeway GA-400) are shown. In the plot, one can see that the standard deviation is comparable to the mean of speed in quite wide range, indicating the existence of non-ignorable variability in speed-density relation. Moreover, the shape of the standard deviation is somewhat interesting, exhibiting a peak around 50 *veh/mi*. Our investigation begins from these observations.



**FIGURE 1 First and second order speed-density relation at one typical site based on GA400 ITS data**

The LWR model assumes that the speed-density ( $v$ - $k$ ) relation is time independent, i.e., system equilibrium is already achieved (by equilibrium, we mean only transition between stationary traffic state is possible). The LWR model says that the density  $k(x, t)$  of traffic flow is the solution to the following equations:

$$\begin{aligned} k_t + (kv)_x &= 0 \\ v &= v(k) \\ k(x, 0) &= k_0(x). \end{aligned} \quad (1)$$

The first equation is the conservation law with the assumption of smoothness of involved functions up to a certain order, the second equation is the fundamental relation which holds under the equilibrium assumption, and the last equation provides the initial state of the solution. Flow  $q = q(k) \equiv kv(k)$  is known as the flux function of  $k$ , which depicts the dynamics of the concerned quantity, i.e., traffic flow density.

To correctly incorporate the exhibited randomness into Equation (1), we first need to analyze the possible sources of randomness. We begin by categorizing the involved randomness in two classes. The first type of randomness is the uncertainty during data collection and processing, e.g., inaccurate reading and data roundoff that is reflected in the initial data setting and scatter plot of the  $v$ - $k$  relation. In this case, the collective small additive errors would altogether obey the normal distribution law, due to the central limit theorem (CLT); alternatively, if the errors are small and multiplicative, they jointly behave by the lognormal law. This type of randomness is statistical and relatively well known. The second type of randomness is due to the inherent system dynamics. Taking the transportation system for example, the drivers' behaviors vary from one driver to another, thus the group is best described in distributional terms rather than deterministically. We claim that this randomness underlies the random  $v$ - $k$  relation as aforementioned. An intuitive interpretation of Figure 1 would be in an analogous manner to that of Brownian bridge. That is, taking the  $v$  as a random process indexed by  $k$ , at two points 0 and jam density  $k_{jam}$  the knowledge of  $v$  is relatively complete since the constraints are imposed by definition of the two states. This explains the smaller variances at two ends as shown. Based on the above analysis, we argue that the second type of randomness is essential, since the first type can usually be controlled reasonably well, for example, through improving the measuring techniques.

Back to the problem of formulating the LWR model in a stochastic setting, we consider the second type randomness since it is dominant and inherent. Most generally, to account for this randomness, we express traffic speed as a positive valued multivariate function,

$$v = v(k, \omega(x, t)) : (\mathbb{R}^+, \mathbb{R}, \mathbb{R}^+, \Omega) \mapsto \mathbb{R}^+, \quad (2)$$

where  $\omega$  is an appropriately defined set on  $\Omega$ , the probability space equipped with measure  $P_{x,t}(\cdot)$ . This definition will lead to a stochastic flux function,

$$f \equiv kv = kv(k, x, t, \omega) : (\mathbb{R}^+, \mathbb{R}, \mathbb{R}^+, \Omega) \mapsto \mathbb{R}^+. \quad (3)$$

Substituting the Formula (3) back into Equation (1), we obtain the stochastic formulation of the LWR as,

$$K_t + (KV(K, \omega))_x = 0. \quad (4)$$

Equation (4) in general admits a random field as solution. Before attempting to obtain useful calculation results, it is desirable to justify the validity of this equation. The minimum requirement is that with trivial probability measure it is consistent with usual deterministic equation. A detailed discussion of the general form of Equation (4) is beyond the scope of current paper.

## 2.2 The FD driven by random free flow speed

To make the analysis and computation tractable, we need restrict our focus to a specific form of (4) in this paper. This is equivalent to treat a specific form of Equation (3). In particular, we assume that the stochastic flux function is independent of space and time, i.e.,

$$f = f(k, \omega). \quad (5)$$

The assumption in the form of (5) greatly simplifies the matter since each realization of  $f$  would be a function of one variable  $k$  only. This reduces to the deterministic case we usually deal with.

To address the features observed in Figure 1, we find the following form plausible. We postulate that the randomness of  $f(\cdot)$  is due to the uncertainty of the free flow speed. First, the free flow speed is written as,

$$v_f \equiv \bar{v}_f + (sk + r)\epsilon \quad (6)$$

where  $\bar{v}_f$  is a constant, and  $\epsilon$  represents the imperfect knowledge of actual  $v_f$ , with  $(sk + r)$  being a scaling coefficient. Adopting this scaling coefficient implicitly assumes that  $v_f$  depends on density  $k$ . Without loss of generality, we assume that  $E\epsilon = 0$  and  $\sigma_\epsilon = 1$  (otherwise, one could let  $\epsilon = (\epsilon - E\epsilon)/\sigma_\epsilon$ ). The reason of using a function of  $k$  as the scaling coefficient is as follows. If we adopt the  $v$ - $k$  relation of,

$$\left(\frac{v(k)}{v_f}\right)^\alpha + \left(\frac{k}{k_{jam}}\right)^\beta = 1, \quad (7)$$

then after substituting Equation (6) in, we have,

$$\begin{aligned} v(k) &\stackrel{d}{=} v_f \left(1 - \left(\frac{k}{k_{jam}}\right)^\beta\right)^{1/\alpha} \\ &= (\bar{v}_f + (sk + r)\epsilon) \left(1 - \left(\frac{k}{k_{jam}}\right)^\beta\right)^{1/\alpha}. \end{aligned} \quad (8)$$

The meaning of  $sk + r$  is interpreted as follows. Letting  $s > 0$ , then as  $k$  increases, the variance of the first term in the product of formula (8) becomes larger. This reflects the common perception that the free flow speed is less informative for the inference of  $v(k)$  when density  $k$  increases, as in this case, the system state represented by  $(k, v)$  is moving away from the state  $(0, v_f)$ . Since  $v_f$  and  $k_{jam}$  are symmetric in Equation (7), we may have a similar argument and term similar to  $(sk + r)\epsilon$  accompanying  $k_{jam}$ , such as  $(\tilde{s}v + \tilde{r})\tilde{\epsilon}$ . However, this will make it difficult to explicitly express  $v$  as a function of  $k$ . Though we can numerically solve  $v$  for a given  $k$  by utilizing an iterative procedure, we would rather put this issue aside at the moment and restrict the focus on the formula (8) to obtain some closed-form results.

From formula (8), taking the expectation at both sides, we obtain,

$$E(v(k)) = \bar{v}_f \left(1 - \left(\frac{k}{k_{jam}}\right)^\beta\right)^{1/\alpha}, \quad (9)$$

and by calculating the second-order moment, we obtain,

$$Var(v(k)) = (sk + r)^2 \left(1 - \left(\frac{k}{k_{jam}}\right)^\beta\right)^{2/\alpha}. \quad (10)$$

when combined with the definition of flux, we get,

$$f(k) \stackrel{d}{=} k(\bar{v}_f + (sk + r)\epsilon) \left(1 - \left(\frac{k}{k_{jam}}\right)^\beta\right)^{1/\alpha}. \quad (11)$$

While the  $v$ - $k$  relation depicted by formula (8) has a lack of sound proof at microscopic level, it has three advantages. First, it is constructed in a heuristic manner as previously mentioned. Its interpretation is quite straightforward and understandable. Second, the postulated  $v$ - $k$  relation is consistent with the observation that a peak of standard deviation exists between 0 and  $k_{jam}$ , which also takes a maximum value in this region. Third, this relation leads to an easy-to-sample random flux function (11), which is actually a family of curves governed by only one random parameter  $\epsilon$ .

At last, we provide the following properties that are useful in the succeeding development of the numerical scheme,

1. For each  $\epsilon$ , function  $f(k)$  is smooth.
2. Upper bound of absolute value of the derivative:

$$\sup_{k \in [k_{jam}, k_0]} |f'(k)| \leq 5k_{jam}|\epsilon_s| + 3|\epsilon_r| + 3\bar{v}_f \quad (12)$$

This is obtained by utilizing the triangle inequality. For technical convenience, we take  $\epsilon \sim U(-\sqrt{3}, \sqrt{3})$ , making the right hand side of (12) a finite value, denoted as  $\alpha_f$ ;

3. There exists decomposition of  $f(k)$  (namely, flux splitting):

$$f(k) = f^+(k) + f^-(k) \quad (13)$$

where  $f^+(k) = (f(k) + \alpha_f k)/2$  and  $f^-(k) = (f(k) - \alpha_f k)/2$ , satisfying  $df^+(k)/dk > 0$  and  $df^-(k)/dk < 0$ .

### 3 Numerical study

#### 3.1 Numerical scheme

In this paper we adopt the finite difference methods with an essentially non oscillatory (ENO) reconstruction to obtain the numerical solution of (1). Roughly speaking, the ENO method reconstructs the cell boundary values through adaptively utilizing the local stencil information. In particular, the stencil with the minimal non-smoothness measure is selected. With ENO, high order finite volume or finite difference methods are immediately available. The order of accuracy depends on the size of the adopted stencil. For a detailed description of this method, the reader is referred to [25] and references therein.

In summary, we devise the algorithm as follows to repeatedly generate random flux function and solve the corresponding LWR model. This algorithm consists of two parts, i.e., initialization and time marching,

##### \* Initialization:

1. Load initial data  $\{k_i^0, i = 1, \dots, N\}$ , where  $k_i^0 = k_0(x_i)$  is the grid point value. Moreover, set  $s = 1, \epsilon_0 = 1$ ;
2. Generate  $\epsilon_s$ , independent of  $\epsilon_1, \dots, \epsilon_{s-1}$  and follows  $U(-\sqrt{3}, \sqrt{3})$ . Generate the random flux function  $f(k)$  following (11);
3. Split the flux function  $f(k) = f^+(k) + f^-(k)$ , following (13);

##### \* Time Marching:

4. Identify  $\{f^+(k_i^n), i = 1, \dots, N\}$  as cell averages and obtain  $v_{i+1/2} = \hat{f}_{i+1/2}^+$  by ENO reconstruction. Similarly, get  $v_{i+1/2}^+ = \hat{f}_{i+1/2}^+$ ;

5. If  $n + 1 \leq T/\Delta t$ , update the point value,

$$k_i^{n+1} = k_i^n - \frac{\Delta t}{\Delta x} (\hat{f}_{i+1/2} - \hat{f}_{i-1/2}) \quad (14)$$

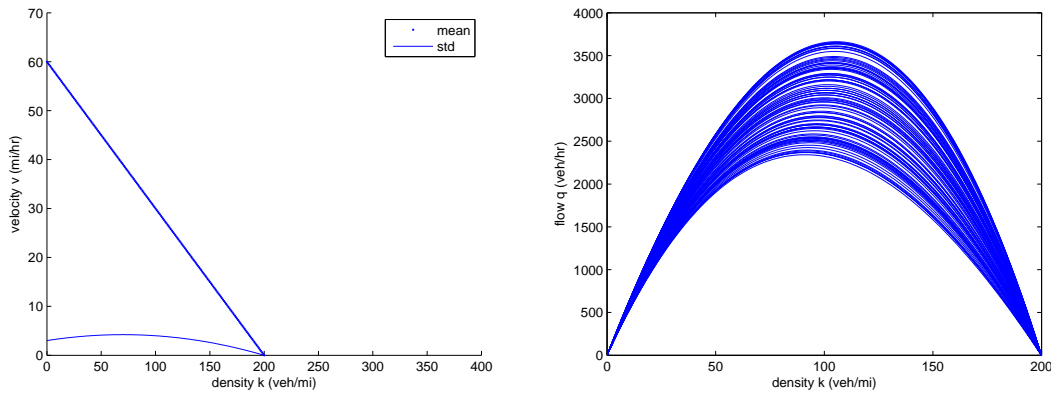
Let  $s = s + 1$ , and go back to 2; else, stop.

### 3.2 Example

To make the model more realistic, the four parameters,  $\alpha$ ,  $\beta$ ,  $s$ , and  $r$  in Equation (11) need to be estimated. In this paper, we adopt the values  $\alpha = 1$ ,  $\beta = 1$ ,  $s = 0.05$  and  $r = 3$  for the purpose of illustration. After setting  $\bar{v}_f = 60 \text{ mi/hr}$  and  $k_{jam} = 200 \text{ veh/mi}$ , the random flux function now takes the form of

$$f(k) \stackrel{d}{=} k(60 + (0.05k + 3)\epsilon)(1 - \frac{k}{200}) \quad (15)$$

Moreover, the free boundary conditions are imposed throughout the simulations in this section, i.e.,  $\partial k/\partial x|_{L,R} = 0$ , where  $L, R$  indicates the left and right boundaries of the computation region. We set  $\Delta t = 1 \text{ sec}$ ,  $\Delta x = 0.1 \text{ mi}$ . The  $v$ - $k$  relation and 20 realizations of the flux in Equation (15) are shown in Figure 2.



**FIGURE 2 Hypothetical random speed-density relation (left panel) and 100 realizations of corresponding random flux function (right panel)**

We first let  $\epsilon = 0$ , which reduces the model depicted by Equation (15) to a deterministic model. With the initial conditions,

$$\begin{aligned} \text{Initial condition a: } (k_l, k_r) &= (110, 30) \\ \text{Initial condition b: } (k_l, k_r) &= (30, 110), \end{aligned} \quad (16)$$

rarefaction wave and shock wave are observed respectively, which are shown in Fig. 3. The solutions predicted with the above flux function and numerical scheme are well expected. In particular, the ENO type finite difference scheme does preserve the shape of shock wave well, with almost no numerical oscillations. Thus we will take the proposed random flux model and employ the

	Mean of $k$	Std of $k$	Mean of $x$	Std of $x$
Under initial condition a	16.09	0.50	5.99	0.75
Under initial condition b	26.46	1.92	8.63	0.63

**TABLE 1 The statistic of numerical solution under initial condition a and b.**

ENO numerical scheme to investigate the case with i.i.d. (independent and identically distributed) random variables  $\epsilon_1, \dots, \epsilon_s$  in a series of simulations.

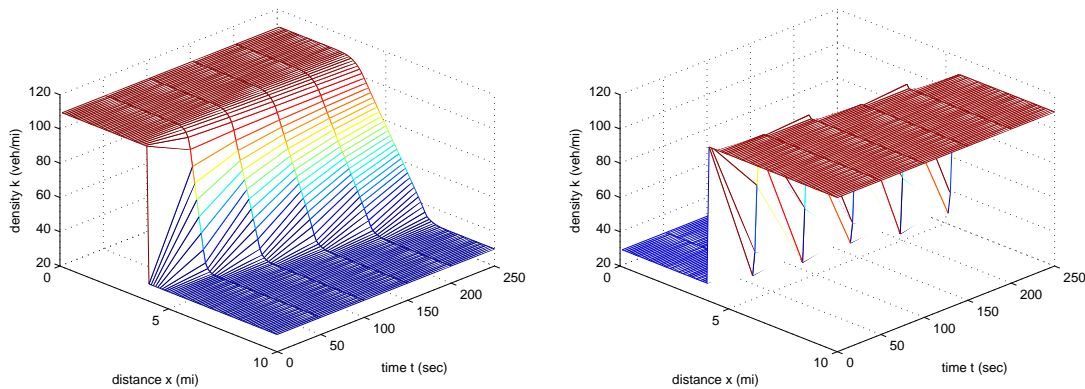
We then investigate the distributional properties of the solution of the proposed model, driven by the i.i.d. random sequence. In particular, the propagation of local disturbance on a one-lane freeway is studied. Here jam/vacuum is defined by significantly different values of local density compared to its neighborhood. We assume  $\epsilon_i \sim U(-\sqrt{3}, \sqrt{3})$  for the aforementioned reason. The settings for the cases studied are as follows,

1. Local jam:

$$k(x, 0) = 50 + 80 \sin((x - 2)\pi)I(2 \leq x \leq 3) \tag{17}$$

2. Local vacuum:

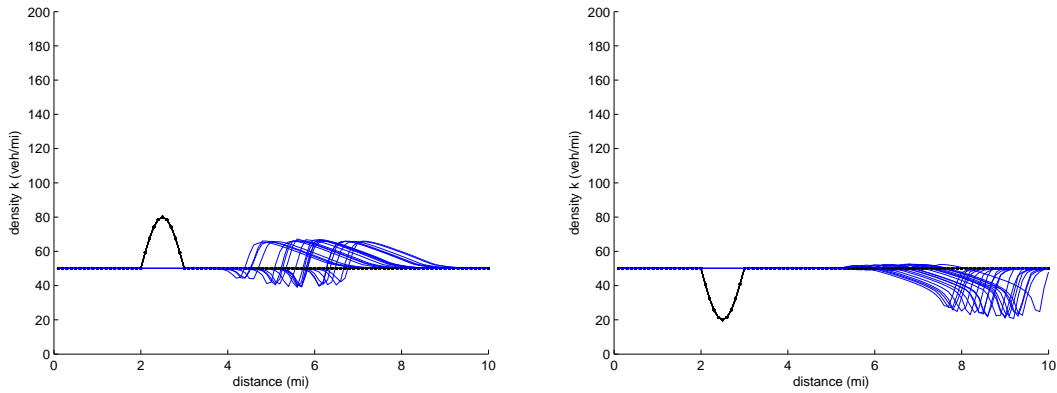
$$k(x, 0) = 50 - 30 \sin((x - 2)\pi)I(2 \leq x \leq 3) \tag{18}$$



**FIGURE 3 Temporal development of density  $k$  with  $\epsilon = 0$ : rarefaction waves (left panel) under Riemann initial condition  $(k_l, k_r) = (110, 30)$ , and shock waves (right panel) under Riemann initial condition  $(k_l, k_r) = (30, 110)$ .**

The simulation results at  $t = 600 \text{ sec}$  are shown in Figure 4. Clearly the predictability of the output drops significantly when the initial randomness has a variance of one. We also list certain statistics of the solutions in Table 1. Two quantities, the maximum local fluctuation and its location, are particularly interesting. The former is defined as,

$$k = \max_{0 \leq x_i \leq 10} |k(x_i) - 50|, \tag{19}$$



**FIGURE 4 Propagation of local disturbance (jam: left panel; vacuum: right panel) with random flux function,  $t = 600$  sec, 20 realizations. The dotted line is initial data and solid lines are simulation results.**

and the latter is defined by,

$$x = \operatorname{argmax}_{x_i} |k(x_i) - 50|. \quad (20)$$

We also calculate the coefficient of variation (CoV) using the values listed in Table 1 and take its reciprocal as the measure of predicting accuracy. It turned out that  $CoV_{a,k_*} > CoV_{b,k_*}$  and  $CoV_{a,x_*} > CoV_{b,x_*}$ .

There are multiple implications from the above results. First, the LWR model is often used to predict the location of shock waves. Our example indicates that caution should be taken with such application when uncertainty of FD is present, because even with small randomness of FD the final outputs (i.e., the density profile) may distribute quite differently. Second, we observe the no trend of decay for the spread of model outputs. This confirms the general intuition that a model deteriorates when applied for long time prediction. Third, quantitatively, the propagation of local vacuum seems more predictable than local jam, while the latter will be of more interest in application. To sum up, our observations are consistent with the view point that LWR model produces qualitatively good prediction[4], but we emphasize that this statement is credible only if the specification of FD is certain and relatively short time scale is involved.

## 4 Concluding remarks

In this paper, motivated by empirical observations from freeway traffic data and noticing the long time debate over FD, we investigate the LWR model in a stochastic setting. In particular, we discuss the general concept of uncertainty and analyze the sources of randomness associated with FD. A specific form of the stochastic flux function is postulated, which is driven by random free flow speed. We provide the mathematical properties of the flux function that are essential for developing a flux splitting scheme. An ENO finite difference numerical scheme is devised and implemented to solve the proposed model. With a hypothetical example, we illustrate how the predictability of two types of local density disturbance travel can be evaluated. This example implies that that if FD cannot be undoubtedly specified, the LWR model will deteriorate and only

1  
2 make reasonable prediction in relatively short time scale. The analysis framework in this article is  
3 hopefully extended to investigation of the general KW models.

4 We remark that the model parameters need to be estimated in order to achieve realistic  
5 simulation results, which is a gap left by the current work. Also, some assumptions are possibly  
6 relaxed in later study. In this sense, our work in this paper is illustrative in nature. Merit of this  
7 study is to initiate the investigations of influences of inherent randomness on traffic modeling. We  
8 expect our study will be further advanced with availability of data of finer resolution in the future.  
9

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