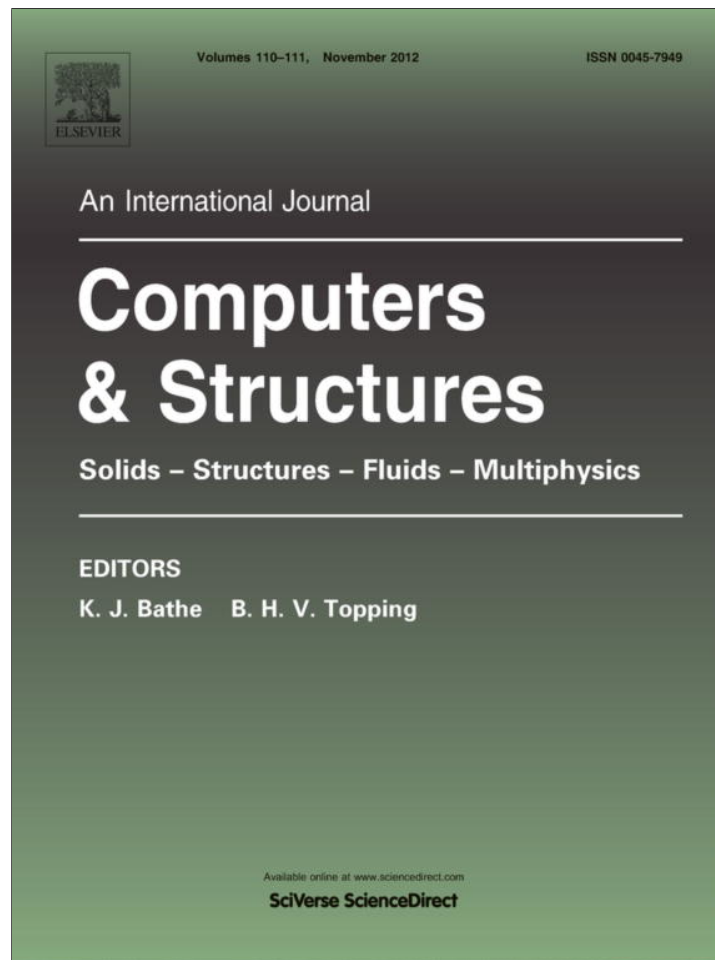


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Stochastic variability of effective properties via the generalized variability response function

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ABSTRACT

Homogenization of randomly heterogeneous material properties into effective properties is an essential procedure in facilitating the analysis of a wide range of mechanics problems. Although formulas exist to calculate deterministic effective properties for structures larger than the representative volume element (RVE), no general method other than Monte Carlo simulation exists to evaluate the variability of these effective properties for structures smaller than the RVE. In a recent paper [1], a method was proposed for evaluating the stochastic variability of effective properties by incorporating the variability response function (VRF) concept. Subsequently, the existence of the VRF for effective properties for linear, statically determinate structures was formally proven. The concept of the VRF has been proposed as a means to systematically capture the effect of the spectral characteristics of uncertain system parameters modeled by homogeneous stochastic fields on the uncertain structural response. Although the existence of VRFs can be formally proven only for statically determinate structures, a Monte Carlo-based methodology has been proposed recently as a generalization of the VRF concept [19]. In this paper, this methodology is extended to establish estimates of the VRF for effective properties of statically indeterminate beams.

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1. Introduction

Homogenization of material properties into effective properties occurs, often implicitly, when conducting standard tests such as tensile tests, direct shear tests, v-notch tests, creep tests, and others. This is because most materials exhibit random heterogeneity at the meso- and microscales. Much work has been done to establish bounds on these effective properties. The most commonly cited pioneering scientific inquiries concerning bounds on effective properties are the works of Voigt and Reuss, both published in German in 1889 and 1929, respectively, from which the famous Reuss and Voigt bounds come. The Voigt bound, also called the isostrain average, gives the upper bound of the elastic modulus. The Reuss bound, called the isostress average, establishes the lower bound of the elastic modulus. Both the Voigt and Reuss bounds can be derived by requiring that the elastic strain energy stored in a material volume is equal when the heterogeneous material is replaced by the homogeneous material and all boundary conditions are kept the same.

It is shown in [23] that elastic effective properties can be considered as deterministic when the structure considered is sufficiently larger than the correlation length scale of the random

heterogeneities. The size of the structure where the effective properties become deterministic corresponds to that of the representative volume element (RVE). When the structure considered is smaller than the RVE, the effective properties are random variables [3,11,14,16,18,31]. With the growing interest in microscale mechanics, characterizing the probabilistic properties of effective properties has become an important research topic. The development of multiscale finite element analysis has provided a means of propagating material property uncertainty across scales [8,7,15,17,28,30]. A need exists, however, for efficient methods based on sound mechanics that provide a direct connection between material property uncertainty at different scales. The goal of this paper is to describe one such method.

The difficulties in establishing detailed probabilistic information (e.g. spectral density function (SDF) and probability distribution function (PDF)) of uncertain parameters in structural systems are due to a lack of data, an inability to measure the desired parameters, model error, noisy measurements, and many more factors. An efficient way to address this problem is to establish functions providing probabilistic characteristics of a structural response quantity while being independent of the uncertain parameters. One such function is the variability response function (VRF), first introduced by in [24], which is essentially a Green's function relating the variance of a response quantity of a structure (e.g. displacement) to the SDF of the uncertain input parameters

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(usually the material properties) that are modeled as random fields.

The applicability of the VRF concept for effective properties is explored for statically indeterminate beams in this paper. The VRF for the effective flexibility in linear, statically determinate beams was introduced in [1]. For linear and nonlinear statically indeterminate beams, a generalized variability response function (GVRF) methodology has been developed to establish generalized variability response functions (GVRFs) for displacement response [19,27]. In this paper, the GVRF methodology is expanded to establish GVRFs for effective flexibility of linear statically indeterminate beams. The paper is outlined as follows. First, the VRF concept is introduced. Then an energy based approach to determine effective properties is presented, followed by the derivation of the VRF for the effective flexibility of a linear statically determinate beam. The GVRF methodology for effective properties is then described in detail and a numerical example is provided to demonstrate its usage. Finally, the results of the example as well as the applicability of the GVRF methodology for effective properties are discussed and evaluated.

2. Variability response function (VRF) concept

In the VRF approach, it is assumed that the uncertain system parameters (e.g. material properties) can be described by a statistically homogeneous random field, $f(x)$. If the displacement response (e.g. $u(x)$ as the transverse displacement of a beam) is the quantity of interest, its variance can be expressed using the following integral form involving the VRF

$$\text{Var}[u(x)] = \int_{-\infty}^{\infty} \text{VRF}_u(x, \kappa) S_f(\kappa) d\kappa, \quad (1)$$

where $S_f(\kappa)$ is the two sided SDF of the zero mean, homogeneous random field $f(x)$ modeling the uncertain system parameters, $u(x)$ is the displacement response of the structure at location x , and κ is the wave number. It should be noted that Eq. (1) is formulated for $f(x)$ being a one-dimensional random field. The VRF is a deterministic function that depends on deterministic properties of the structure, loading, and boundary conditions. It identifies the sensitivity of the response variability to the spectral characteristics (or equivalently the correlation structure) of $f(x)$ and provides the supremum of the response variance if only the variance of $f(x)$ is known. There are a few exact analytical derivations of VRFs for statically determinate structures [1,2,6,27]. Through numerical techniques, however, numerous approximations of the VRF for statically indeterminate structures have been made [9,10,20–22,25,29].

A computationally efficient numerical approach called the Fast Monte Carlo method, first proposed in [25] and further developed in [21,22], can be applied to general mechanical/structural systems to establish an approximation of the VRF of statically indeterminate structures. This method involves a fundamental conjecture: a unique VRF exists for statically indeterminate structures that is independent of the PDF and SDF of the stochastic field modeling the uncertain system parameters. In order to validate this conjecture, a methodology has been proposed as a generalization of the Fast Monte Carlo method, the aim of which is to establish a generalized VRF (GVRF) for statically indeterminate structures [19,27] without having to assume the existence of Eq. (1) a priori. According to the GVRF methodology, the uncertain system parameters are described by stochastic fields with a wide range of combinations of marginal PDFs and SDFs. For each combination considered, a corresponding GVRF is computed. If all the computed GVRFs are approximately the same (allowing for numerical and estimation errors), then it can be claimed that an approximate VRF exists that is nearly

independent from the SDF and the PDF. This methodology is described in detail in Section 5.

In a recent paper [1], the VRF for the effective flexibility of heterogeneous statically determinate structures was analytically derived. The formulation is an extension of the VRF concept for the displacement response of statically determinate structures [6]. In this case, a VRF was derived for the variance of the effective flexibility \bar{D} , which depends on the displacement response field $u(x)$, as:

$$\text{Var}[\bar{D}] = \int_{-\infty}^{\infty} \text{VRF}_{\bar{D}}(\kappa) S_f(\kappa) d\kappa, \quad (2)$$

where $S_f(\kappa)$ is the two sided SDF of the zero mean, homogeneous random field $f(x)$ modeling the fluctuations of the flexibility about its mean value. A general property of the VRF is that as $\kappa \rightarrow +\infty$, $\text{VRF} \rightarrow 0$ because extremely rapid fluctuations (equivalently fluctuations of very small wave length) are not felt by the structure. If the VRF equals zero for a given range of wave numbers, then a random material property having all of its power within this range of wave numbers will produce an effective property that is deterministic (constant) and not random (i.e. the effective property will be that of the RVE).

3. Effective properties

Consider a heterogeneous body Ω described by coordinates $\mathbf{x} \in \mathfrak{R}^3$ whose material properties can be described as locally isotropic. The strong form of the boundary value problem with its boundary conditions is

$$\sigma_{ij,j} + \mathbf{b}_i = 0 \quad (3a)$$

$$\sigma_{ij} = \mathbf{C}_{ijkl}(\mathbf{x}) \epsilon_{kl} \quad (3b)$$

$$\epsilon_{ij} = \frac{1}{2} (\mathbf{u}_{i,j} + \mathbf{u}_{j,i}) \quad (3c)$$

$$\sigma_{ij} \mathbf{n}_j = \bar{\mathbf{t}}_i \in \Gamma_t \quad (4a)$$

$$\mathbf{u}_i = \bar{\mathbf{u}}_i \in \Gamma_u \quad (4b)$$

$$\Gamma_t \cup \Gamma_u = \partial\Omega \quad \text{and} \quad \Gamma_t \cap \Gamma_u = \emptyset, \quad (4c)$$

where σ and ϵ are the stress and strain tensors, respectively, while \mathbf{u} and \mathbf{b} are the displacement and body force vectors, respectively. The boundary $\partial\Omega$, defined by the outward unit normal vector, \mathbf{n} , is the union of spaces Γ_t and Γ_u defining the spaces of prescribed traction, $\bar{\mathbf{t}}$, and prescribed displacement, $\bar{\mathbf{u}}$, respectively. The constitutive tensor $\mathbf{C}(\mathbf{x})$ is a function of position \mathbf{x} due to random fluctuations of the elastic modulus or Poisson's ratio of the material occupying Ω . Let a homogeneous counterpart of Ω , denoted Ω_H , be occupied by a material with a constitutive tensor, $\bar{\mathbf{C}}$, that is constant within Ω_H but is a function of the displacement boundary conditions (Eq. (4b)), surface tractions (Eq. (4a)), and an integral expression of $\mathbf{C}(\mathbf{x})$. The effective material properties are determined such that the strain energy in Ω_H equals the strain energy in Ω under the same set of boundary conditions, that is

$$\begin{aligned} \frac{1}{2} \int_{\Omega_H} \epsilon_0(\mathbf{x}) \cdot \bar{\mathbf{C}} \cdot \epsilon_0(\mathbf{x}) dV &= \frac{1}{2} \int_{\Omega} \epsilon(\mathbf{x}) \cdot \mathbf{C}(\mathbf{x}) \cdot \epsilon(\mathbf{x}) dV \\ &= \int_{\Gamma_t} \mathbf{u}(\mathbf{x}) \bar{\mathbf{t}}(\mathbf{x}) d\Gamma_t, \end{aligned} \quad (5)$$

where $\epsilon_0(\mathbf{x})$ is the strain in Ω_H , and ' \cdot ' denotes the tensor inner product.

Consider now the case where Poisson's ratio is constant and only the elastic modulus $E(\mathbf{x})$ is randomly heterogeneous. Then the effective elastic modulus \bar{E} can be factored out of the effective constitutive tensor (i.e. $\bar{\mathbf{C}} = \bar{E} \bar{\mathbf{C}}'$), and can be expressed as

$$\bar{E} = \frac{\int_{\Gamma_t} \mathbf{u}(\mathbf{x}) \bar{\mathbf{t}}(\mathbf{x}) d\Gamma_t}{\frac{1}{2} \int_{\Omega_H} \boldsymbol{\epsilon}_0(\mathbf{x}) \cdot \bar{\mathbf{C}}' \cdot \boldsymbol{\epsilon}_0(\mathbf{x}) dV} = \frac{\int_{\Omega} \boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{C}(\mathbf{x}) \cdot \boldsymbol{\epsilon}(\mathbf{x}) dV}{\int_{\Omega_H} \boldsymbol{\epsilon}_0(\mathbf{x}) \cdot \bar{\mathbf{C}}' \cdot \boldsymbol{\epsilon}_0(\mathbf{x}) dV}. \quad (6)$$

The effective elastic modulus is bounded by the Reuss and Voigt bounds [13]

$$\bar{E}_r \leq \bar{E} \leq \bar{E}_v \quad (7a)$$

$$\text{Reuss : } \bar{E}_r = \frac{1}{V_{\Omega}} \left[\int_{\Omega} E(\mathbf{x})^{-1} dV \right]^{-1} \quad \text{Voigt : } \bar{E}_v = \frac{1}{V_{\Omega}} \int_{\Omega} E(\mathbf{x}) dV \quad (7b)$$

From Eq. (6), the variance of the effective elastic modulus is determined as

$$\text{Var}[\bar{E}] = \frac{1}{C_1^2} \text{Var} \left[\int_{\Gamma_t} \mathbf{u}(\mathbf{x}) \bar{\mathbf{t}}(\mathbf{x}) d\Gamma_t \right], \quad (8)$$

where $C_1 = \frac{1}{2} \int_{\Omega_H} \boldsymbol{\epsilon}_0(\mathbf{x}) \cdot \bar{\mathbf{C}}' \cdot \boldsymbol{\epsilon}_0(\mathbf{x}) dV$. If the random fluctuations of the elastic modulus around its mean value are described by a statistically homogeneous, zero mean random field, $f(\mathbf{x})$, then the goal of the VRF concept is to establish a relationship of the following form

$$\text{Var}[\bar{E}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_f(\kappa_1, \kappa_2, \kappa_3) \text{VRF}_{\bar{E}}(\kappa_1, \kappa_2, \kappa_3) d\kappa_1 d\kappa_2 d\kappa_3, \quad (9)$$

where $S_f(\kappa_1, \kappa_2, \kappa_3)$ is the SDF of $f(\mathbf{x})$. Only for cases involving statically determinate structures can Eq. (9) be proven to exist, because for statically indeterminate structures the displacement cannot be expressed in general by a function that is separable with respect to the applied traction and the stochastic parameters of the constitutive law.

3.1. Effective flexibility for beams

It is more convenient to describe the constitutive law in terms of the flexibility for beam structures, that is

$$D(x) = \frac{1}{E(x)I} = \frac{1}{E_0 I} (1 + f(x)), \quad (10)$$

where E_0 is the mean value of $E(x)$ and I is the moment of inertia of the beam.

Similarly, a homogeneous beam is established such that the external work \mathbb{W} for the heterogeneous beam due to a distributed load $q(x)$, concentrated load P , and concentrated moment M is equal to that on the homogeneous beam \mathbb{W}_H under the exact same loading. The expressions for \mathbb{W} and \mathbb{W}_H are given by

$$\mathbb{W} = \int_0^L u(x)q(x)dx + Pu(x_p) + M\theta(x_M) \quad (11a)$$

$$\mathbb{W}_H = \bar{D} \left[\int_0^L u_0(x)q(x)dx + Pu_0(x_p) + M\theta_0(x_M) \right] = C_1 \bar{D}, \quad (11b)$$

where x_p is the coordinate along the length of the beam where the concentrated load P is applied, and x_M is the coordinate where the concentrated moment M is applied.

In Eq. (11a), $u(x)$ is the displacement of the heterogeneous beam and $\theta(x)$ is the slope of the displacement of the heterogeneous beam (i.e. $\theta(x) = \frac{du(x)}{dx}$). In Eq. (11b), \bar{D} is the effective flexibility, $u_0(x) = 1/\bar{D}u_H(x)$ and $\theta_0(x) = 1/\bar{D}\theta_H(x)$ with $u_H(x)$ and $\theta_H(x)$ being the displacement and slope of the homogeneous beam. Eqs. (11a) and (11b) can be easily extended to account for multiple applied concentrated loads and moments. Combining eventually Eqs. (11a) and (11b), the effective flexibility is defined as

$$\bar{D} = \frac{1}{C_1} \left[\int_0^L u(x)q(x)dx + Pu(x_p) + M\theta(x_M) \right] \quad (12)$$

4. Derivation of VRF for effective flexibility for statically determinate beam structures

The following derivation is an extension of the derivation in [1]. For a statically determinate beam, the displacement and its derivative can be written as

$$u(x) = \int_0^x G_u(x, s) \frac{m(s)}{E_0 I} (1 + f(s)) ds, \quad (13a)$$

$$\theta(x) = \int_0^x G_{\theta}(x, s) \frac{m(s)}{E_0 I} (1 + f(s)) ds, \quad (13b)$$

where $G_u(x, s)$ and $G_{\theta}(x, s)$ are the Green's functions of the displacement and the slope of the displacement, respectively, associated with the governing differential beam equation, and $m(s)$ is the internal moment distribution along the length of the beam. Taking into account that C_1 is a deterministic constant, the expected value of the effective flexibility can be written as

$$\begin{aligned} C_1 \mathbb{E}[\bar{D}] &= \mathbb{E} \left[\int_0^L q(x) \int_0^x G_u(x, s) \frac{m(s)}{E_0 I} (1 + f(s)) ds dx + P \int_0^{x_p} G_u(x_p, s) \frac{m(s)}{E_0 I} (1 + f(s)) ds + M \int_0^{x_M} G_{\theta}(x_M, s) \frac{m(s)}{E_0 I} (1 + f(s)) ds \right] \\ &= \int_0^L q(x) \int_0^x G_u(x, s) \frac{m(s)}{E_0 I} ds dx + P \int_0^{x_p} G_u(x_p, s) \frac{m(s)}{E_0 I} ds \\ &\quad + M \int_0^{x_M} G_{\theta}(x_M, s) \frac{m(s)}{E_0 I} ds \end{aligned} \quad (14)$$

and the mean square as

$$\begin{aligned} C_1^2 \mathbb{E}[\bar{D}^2] &= \int_0^L \int_0^L q(x_1)q(x_2) \int_0^{x_1} \int_0^{x_2} G_u(x_1, s_1)G_u(x_2, s_2) \\ &\quad \times \frac{m(s_1)m(s_2)}{(E_0 I)^2} \mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 dx_2 dx_1 \\ &\quad + 2P \int_0^L q(x) \int_0^x \int_0^{x_p} G_u(x, s_1)G_u(x_p, s_2) \\ &\quad \times \frac{m(s_1)m(s_2)}{(E_0 I)^2} \mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 dx \\ &\quad + 2M \int_0^L q(x) \int_0^x \int_0^{x_M} G_u(x, s_1)G_{\theta}(x_M, s_2) \\ &\quad \times \frac{m(s_1)m(s_2)}{(E_0 I)^2} \mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 dx \\ &\quad + P^2 \int_0^{x_p} \int_0^{x_p} G_u(x_p, s_1)G_u(x_p, s_2) \\ &\quad \times \frac{m(s_1)m(s_2)}{(E_0 I)^2} \mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 \\ &\quad + 2PM \int_0^{x_p} \int_0^{x_M} G_u(x_p, s_1)G_{\theta}(x_M, s_2) \\ &\quad \times \frac{m(s_1)m(s_2)}{(E_0 I)^2} \mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 + M^2 \\ &\quad \times \int_0^{x_M} \int_0^{x_M} G_{\theta}(x_M, s_1)G_{\theta}(x_M, s_2) \frac{m(s_1)m(s_2)}{(E_0 I)^2} \mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1. \end{aligned} \quad (15)$$

Combining Eqs. (14) and (15) yields the variance of the effective flexibility

$$\begin{aligned}
 C_1^2 \text{Var}[\bar{D}] = & \int_0^L \int_0^L q(x_1)q(x_2) \int_0^{x_1} \int_0^{x_2} G_u(x_1, s_1)G_u(x_2, s_2) \\
 & \times \frac{m(s_1)m(s_2)}{(E_0I)^2} \mathbb{E}[f(s_1)f(s_2)] ds_2 ds_1 dx_2 dx_1 \\
 & + 2P \int_0^L q(x) \int_0^x \int_0^{x_p} G_u(x, s_1)G_u(x_p, s_2) \\
 & \times \frac{m(s_1)m(s_2)}{(E_0I)^2} \mathbb{E}[f(s_1)f(s_2)] ds_2 ds_1 dx + 2M \\
 & \int_0^L q(x) \int_0^x \int_0^{x_M} G_u(x, s_1)G_\theta(x_M, s_2) \frac{m(s_1)m(s_2)}{(E_0I)^2} \\
 & \times \mathbb{E}[f(s_1)f(s_2)] ds_2 ds_1 dx + P^2 \int_0^{x_p} \int_0^{x_p} G_u(x_p, s_1)G_u(x_p, s_2) \\
 & \times \frac{m(s_1)m(s_2)}{(E_0I)^2} \mathbb{E}[f(s_1)f(s_2)] ds_2 ds_1 + 2PM \int_0^{x_p} \\
 & \int_0^{x_M} G_u(x_p, s_1)G_\theta(x_M, s_2) \frac{m(s_1)m(s_2)}{(E_0I)^2} \mathbb{E}[f(s_1)f(s_2)] ds_2 ds_1 \\
 & + M^2 \int_0^{x_M} \int_0^{x_M} G_\theta(x_M, s_1)G_\theta(x_M, s_2) \\
 & \times \frac{m(s_1)m(s_2)}{(E_0I)^2} \mathbb{E}[f(s_1)f(s_2)] ds_2 ds_1. \tag{16}
 \end{aligned}$$

Using the Wiener–Khinchin transform to replace the autocorrelation function $\mathbb{E}[f(s_1)f(s_2)]$ with the SDF $S_f(\kappa)$, the variance can be written as

$$\text{Var}[\bar{D}] = \int_{-\infty}^{\infty} \text{VRF}_{\bar{D}}(\kappa) S_f(\kappa) d\kappa, \tag{17}$$

where the variability response function for the effective flexibility is given by

$$\begin{aligned}
 C_1^2 \text{VRF}_{\bar{D}}(\kappa) = & \int_0^L \int_0^L q(x_1)q(x_2) \int_0^{x_1} \int_0^{x_2} G_u(x_1, s_1)G_u(x_2, s_2) \\
 & \times \frac{m(s_1)m(s_2)}{(E_0I)^2} \exp(i\kappa(s_2 - s_1)) ds_2 ds_1 dx_2 dx_1 \\
 & + 2P \int_0^L q(x) \int_0^x \int_0^{x_p} G_u(x, s_1)G_u(x_p, s_2) \\
 & \times \frac{m(s_1)m(s_2)}{(E_0I)^2} \exp(i\kappa(s_2 - s_1)) ds_2 ds_1 dx \\
 & + 2M \int_0^L q(x) \int_0^x \int_0^{x_M} G_u(x, s_1)G_\theta(x_M, s_2) \\
 & \times \frac{m(s_1)m(s_2)}{(E_0I)^2} \exp(i\kappa(s_2 - s_1)) ds_2 ds_1 dx \\
 & + P^2 \int_0^{x_p} \int_0^{x_p} G_u(x_p, s_1)G_u(x_p, s_2) \frac{m(s_1)m(s_2)}{(E_0I)^2} \\
 & \times \exp(i\kappa(s_2 - s_1)) ds_2 ds_1 + 2PM \\
 & \int_0^{x_p} \int_0^{x_M} G_u(x_p, s_1)G_\theta(x_M, s_2) \frac{m(s_1)m(s_2)}{(E_0I)^2} \\
 & \times \exp(i\kappa(s_2 - s_1)) ds_2 ds_1 + M^2 \\
 & \int_0^{x_M} \int_0^{x_M} G_\theta(x_M, s_1)G_\theta(x_M, s_2) \frac{m(s_1)m(s_2)}{(E_0I)^2} \\
 & \times \exp(i\kappa(s_2 - s_1)) ds_2 ds_1. \tag{18}
 \end{aligned}$$

4.1. Cantilever example

Consider the cantilever in Fig. 1 with length L , distributed loading $q(x) = q_0(L - x)/L$, average flexibility $1/E_0I$, and concentrated loads and moments equal to zero without any loss of generality. The Green's function for the displacement is

$$G(x, s) = \begin{cases} x - s, & s < x \\ 0, & s \geq x. \end{cases} \tag{19}$$

The moment is determined from statics as: $m(x) = -q_0L^2/6 + q_0Lx/2 - q_0x^2/2 + q_0x^3/(6L)$, and the deterministic coefficient C_1 is solved to be: $C_1 = q^2L^5/34.52$. After some algebra, the variance of the effective flexibility is expressed as in Eq. (17) with the VRF given by

$$\begin{aligned}
 \text{VRF}_{\bar{D}}(\kappa) = & \left(\frac{34.52}{q^2L^5} \right)^2 \int_0^L \int_0^L \frac{q_0(L - x_1)q_0(L - x_2)}{L^2} \\
 & \int_0^{x_1} \int_0^{x_2} (x_1 - s_1)(x_2 - s_2) \\
 & \times \frac{\left(-\frac{q_0L^2}{6} + \frac{q_0Ls_1}{2} - \frac{q_0s_1^2}{2} + \frac{q_0s_1^3}{6L} \right) \left(-\frac{q_0L^2}{6} + \frac{q_0Ls_2}{2} - \frac{q_0s_2^2}{2} + \frac{q_0s_2^3}{6L} \right)}{(E_0I)^2} \\
 & \times \exp(i\kappa(s_2 - s_1)) ds_2 ds_1 dx_2 dx_1. \tag{20}
 \end{aligned}$$

The integrals in Eq. (20) are evaluated analytically with the help of MAPLE, and $\text{VRF}_{\bar{D}}(\kappa)$ is plotted in Fig. 2 using the following values for the various parameters: $L = 15$ m, $q_0 = 1000$ N/m, $1/(E_0I) = 3.2 \times 10^{-6}$ 1/N m².

5. GVRF methodology

The existence and SDF/PDF-independence of the VRF for response displacements has been formally proven for linear, statically determinate structures [2,6] and for a special class of nonlinear, statically determinate structures in reference [27]. The existence and SDF/PDF-independence of the VRF for effective flexibility has been formally proven for linear, statically determinate beams in reference [1] and was presented in more detail in Section 4. The VRF's existence and SDF/PDF-independence, however, has never been formally proven for any statically indeterminate structure. This chapter details a Monte Carlo based methodology, proposed in [19], that generalizes the VRF concept so that it can be applied to statically indeterminate linear structures when considering either the response displacements or an effective property. The aim of this methodology is to compute a Generalized VRF (GVRF) with properties essentially identical to those of the classical VRF. This methodology involves the following conjecture: there exists a GVRF for indeterminate structures that is approximately independent of the marginal PDF and SDF of the uncertain system parameters. The main objective of the methodology is a Monte Carlo simulation procedure developed to specifically determine the validity of this conjecture.

5.1. GVRF methodology for effective properties of linear, statically indeterminate structures

For a specific linear statically indeterminate structure with a randomly heterogeneous material property modeled by a specific one-dimensional homogeneous stochastic field $f(x)$ with SDF $S_f(\kappa)$

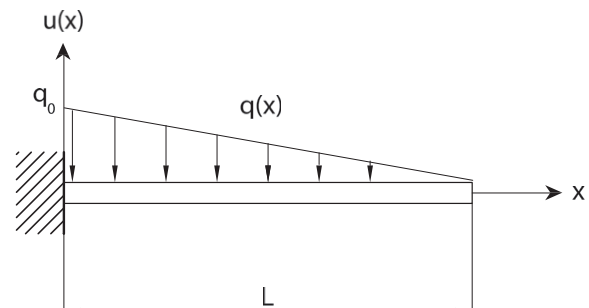


Fig. 1. Cantilever analyzed.

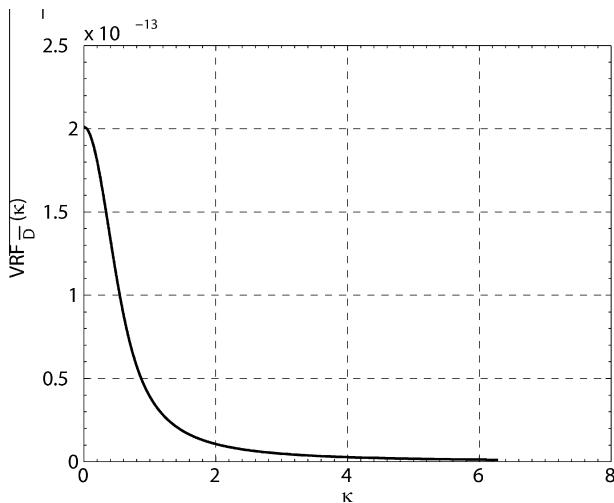


Fig. 2. VRF for effective flexibility of cantilever shown in Fig. 1.

and prescribed marginal PDF, the variance of its effective property \bar{D} can always be written in the following integral form involving some function $G\text{VRF}_{\bar{D}}(\kappa)$ (standing for generalized variability response function)

$$\text{Var}[\bar{D}] = \int_{-\infty}^{\infty} S_f(\kappa)G\text{VRF}_{\bar{D}}(\kappa)d\kappa. \quad (21)$$

The left-hand-side of the above equation, $\text{Var}[\bar{D}]$, can be easily computed through brute-force Monte Carlo simulation by generating sample functions of stochastic field $f(x)$ using its prescribed SDF and PDF. It is obvious, however, that there is no unique solution for $G\text{VRF}_{\bar{D}}(\kappa)$ in Eq. (21) for a given $S_f(\kappa)$. Eq. (21) can be written in discretized form as

$$\text{Var}[\bar{D}] = 2 \sum_{l=1}^N S_f(\kappa_l)G\text{VRF}_{\bar{D}}(\kappa_l)\Delta\kappa, \quad (22)$$

where the wave number domain is discretized into N equal intervals $\Delta\kappa$ between 0 and an upper cutoff wave number κ_u , and the set of wave numbers $\kappa_n = (n - 1)\Delta\kappa + \Delta\kappa/2$ are the center points of the intervals. Eq. (22) can be written equivalently in matrix for as

$$\text{Var}[\bar{D}] = 2[S_f(\kappa_1) \quad S_f(\kappa_2) \quad \dots \quad S_f(\kappa_N)] \begin{bmatrix} G\text{VRF}_{\bar{D}}(\kappa_1) \\ G\text{VRF}_{\bar{D}}(\kappa_2) \\ \vdots \\ G\text{VRF}_{\bar{D}}(\kappa_N) \end{bmatrix} \Delta\kappa. \quad (23)$$

Consider now that N different stochastic fields with N different SDFs are selected (but all N with the same marginal PDF) and that Eq. (23) is written repeatedly for each one of these N fields/SDFs. This leads to a system of N linear equations with N unknowns, where the unknowns are contained in the vector of discretized values of the GVRF. The left-hand-side vector of variances can be easily computed by brute-force Monte Carlo simulations as mentioned earlier. The resulting system is shown below

$$\begin{bmatrix} \text{Var}[\bar{D}_1] \\ \text{Var}[\bar{D}_2] \\ \vdots \\ \text{Var}[\bar{D}_N] \end{bmatrix} = 2 \begin{bmatrix} S_{f_1}(\kappa_1) & S_{f_1}(\kappa_2) & \dots & S_{f_1}(\kappa_N) \\ S_{f_2}(\kappa_1) & S_{f_2}(\kappa_2) & \dots & S_{f_2}(\kappa_N) \\ \vdots & \vdots & \ddots & \vdots \\ S_{f_N}(\kappa_1) & S_{f_N}(\kappa_2) & \dots & S_{f_N}(\kappa_N) \end{bmatrix} \begin{bmatrix} G\text{VRF}_{\bar{D}}(\kappa_1) \\ G\text{VRF}_{\bar{D}}(\kappa_2) \\ \vdots \\ G\text{VRF}_{\bar{D}}(\kappa_N) \end{bmatrix} \Delta\kappa. \quad (24)$$

Each row in Eq. (24) corresponds to a different stochastic field and consequently a different SDF, $S_{f_n}(\kappa)$, $n = 1, 2, \dots, N$. All these N fields have the same marginal PDF. The solution of the system of

N linear equations with N unknowns in Eq. (24) will now provide a unique solution for the vector of discretized values of the GVRF.

The entire process resulting in Eq. (24) is repeated for several other sets of N different fields/SDFs paired with a wide range of different marginal PDFs. If the solutions of all these systems of N linear equations yield approximately the same solution for the GVRF (allowing for small differences due to numerical reasons), then it can be claimed with reasonable certainty that an approximate VRF exists for this structure that it is almost entirely independent of the SDF and the PDF of the stochastic field modeling the uncertain system properties. For more information on this methodology, the reader is referred to [19].

5.2. Non-Gaussian stochastic fields considered

The non-Gaussian stochastic field models considered here for $f(x)$ are either memoryless translation fields [12] (mapped from an underlying Gaussian field) or associated fields [4,5] (mapped from an underlying U-shaped Beta random sinusoid field). The underlying field is denoted by $g(x)$ and the mapped/transformed field by $f(x)$, while the two corresponding marginal cumulative distribution functions are denoted by $P_g(\cdot)$ and $P_f(\cdot)$, respectively. Then, whether $f(x)$ is a translation field or an associated field, it is defined through the following transformation

$$f(x) = P_f^{-1}(P_g(g(x))), \quad (25)$$

when $f(x)$ is a translation field, $g(x)$ is a Gaussian field. When $f(x)$ is an associated field, $g(x)$ is a U-shaped Beta random sinusoid field. The marginal PDFs considered for $f(x)$ include truncated Gaussian, Lognormal, and Uniform distributions.

Realizations of $f(x)$ can be generated by simulating $g(x)$ and then performing the transformation in Eq. (25). In the case of a translation field for $f(x)$, the underlying Gaussian field $g(x)$ is simulated using the Spectral Representation Method outlined in [26]. In the case of an associated field for $f(x)$, the underlying field $g(x)$ is a U-shaped Beta random sinusoid field. The U-shaped beta random sinusoid field has an SDF which consists of a delta function located at wave number κ_δ , given by

$$S_g(\kappa) = \frac{1}{2} \sigma_g^2 [\delta(\kappa - \kappa_\delta) + \delta(\kappa + \kappa_\delta)], \quad (26)$$

while the field itself is expressed as

$$g(x) = \sqrt{2} \sigma_g \cos(\kappa_\delta x + \theta), \quad \theta \text{ Uniform in } [0, 2\pi], \quad (27)$$

with σ_g denoting the standard deviation of $g(x)$. Eq. (27) can be used in a straightforward way to generate sample realizations of $g(x)$.

Three marginal PDFs are considered for the non-Gaussian stochastic field $f(x)$ (whether translation or associated). The expressions for $f(x)$ are given below in terms of $P_g(g(x))$:

Truncated Gaussian (TG)

$$f(x) = \begin{cases} a_l, & s\Phi^{-1}(P_g(g(x))) + m < a_l \\ s\Phi^{-1}(P_g(g(x))) + m, & a_l \leq s\Phi^{-1}(P_g(g(x))) + m \leq a_u \\ a_u, & a_u < s\Phi^{-1}(P_g(g(x))) + m. \end{cases} \quad (28)$$

Uniform (UN)

$$f(x) = (a_u - a_l)P_g(g(x)) + a_l \quad (29)$$

Lognormal (LN)

$$f(x) = \exp(m + s\Phi^{-1}(P_g(g(x)))) + a_l, \quad (30)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the unit normal distribution. One specific case is considered from each one of

Table 1
Parameters of three zero-mean marginal PDFs considered for $f(x)$.

PDF	Parameters				σ_f
	a_l	a_u	m	s	
LN	-.799	-	-.45	$\sqrt{.45}$.60
TG	-.90	.90	0.0	1.0	.67
UN	-.99	.99	-	-	.57

the above three probability distributions and the resulting three cases for the marginal PDFs of $f(x)$ are fully defined in Table 1. It should be noted that all three of these marginal PDFs have zero mean value (their standard deviation is denoted by σ_f and is provided in Table 1).

The cumulative distribution functions of the two underlying fields are given by
Gaussian

$$P_g(g(x)) = \Phi(g(x)) = \int_{-\infty}^{g(x)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds, \quad -\infty < g(x) < \infty \quad (31a)$$

U-Beta

$$P_g(g(x)) = 1 - \frac{1}{\pi} \cos^{-1}\left(\frac{g(x)}{\sqrt{2}}\right) = 1 - \frac{1}{\pi} \cos^{-1}(\cos(\kappa_s x + \theta)), \quad -\sqrt{2} < g(x) < \sqrt{2}. \quad (31b)$$

Note that both of the above cumulative distribution functions have zero mean and unit standard deviation ($\sigma_g = 1$).

5.3. Families of spectral density functions considered

Each row of the matrix in Eq. (24) represents the SDF of one of the N transformed fields $f_n(x)$; $n = 1, 2, \dots, N$. However, it is the underlying fields $g_n(x)$; $n = 1, 2, \dots, N$ that are defined first, and then they are transformed into the corresponding $f_n(x)$; $n = 1, 2, \dots, N$ through the mapping in Eq. (25). The criteria for selecting one family of N underlying fields $g_n(x)$; $n = 1, 2, \dots, N$ from which the entire matrix in Eq. (24) will be eventually built are the following.

1. The SDFs of the N fields $g_n(x)$; $n = 1, 2, \dots, N$ should show as high a diversity as possible in providing power over the entire wave number range considered: $[0, \kappa_u]$.
2. All of these N fields should have the same marginal PDF $P_g(\cdot)$ and consequently the same variance.
3. The N SDFs should be organized in a way such that the condition number of the matrix in Eq. (24) is minimized.

In addition, it should be kept in mind that the N fields $g_n(x)$; $n = 1, 2, \dots, N$ will be transformed into the N fields $f_n(x)$; $n = 1, 2, \dots, N$ using the same marginal PDF $P_f(\cdot)$.

An efficient structure to satisfy the aforementioned conditions is when the SDFs of the N fields $g_n(x)$; $n = 1, 2, \dots, N$ all have the same shape, differing only by repeated shifts of $\Delta\kappa$. In order for all N fields to maintain the same variance, the corresponding SDFs are defined in a circulant manner: as the SDFs are shifted towards the upper cutoff wave number κ_u , the values that would extend beyond κ_u are carried over to the origin of the wave number domain as described by Eq. (32). This circulant structure is demonstrated in Fig. 3 where $\kappa_u = 2\pi$ and $N = 128$. The SDF of $g_1(x)$ (out of the N SDFs $S_{g_n}(\kappa)$; $n = 1, 2, \dots, N$) is known as the parent SDF of this family and is denoted by $S_p(\kappa)$. This parent SDF is shown in Fig. 3(a) for

one of the two cases considered in this study and is denoted by $S_{p_1}(\kappa)$ (exponential decay defined in the first row of Eq. (33)). Five more SDFs are shown from this family in Fig. 3: the 25th, the 60th, the 80th, the 105th and the 122nd (out of a total of 128). The n th SDF of a family $S_{g_n}(\kappa)$ is defined in terms of the parent SDF $S_p(\kappa)$ as

$$S_{g_n}(\kappa) = \begin{cases} S_p(\kappa + \kappa_u - n\Delta\kappa + \Delta\kappa), & 0 \leq \kappa \leq (n-1)\Delta\kappa \\ S_p(\kappa - n\Delta\kappa), & n\Delta\kappa \leq \kappa \leq \kappa_u. \end{cases} \quad (32)$$

Two families of N SDFs are considered in this study. The corresponding parent SDFs are denoted by $S_{p_1}(\kappa)$ and $S_{p_2}(\kappa)$ and are given by

$$\begin{aligned} S_{p_1}(\kappa) &= \exp(-2|\kappa|) \\ S_{p_2}(\kappa) &= \delta(\kappa) \end{aligned} \quad (33)$$

It should be noted that $S_{p_2}(\kappa)$ is used for the underlying U-shaped Beta random sinusoid field, while $S_{p_1}(\kappa)$ is used for the underlying Gaussian field.

The preceding discussion described the definition of a family of N underlying fields $g_n(x)$; $n = 1, 2, \dots, N$ and of their corresponding SDFs. However, it is necessary to determine the SDFs of the transformed fields $f_n(x)$; $n = 1, 2, \dots, N$ so that they can be used to construct the matrix in Eq. (24). The SDFs of $f_n(x)$; $n = 1, 2, \dots, N$ can be computed numerically in a straightforward way using translation field theory [12] or associated field theory [19].

The terminology used to identify a specific family of stochastic fields $f_n(x)$; $n = 1, 2, \dots, N$ associated with the computation of a GVRF is explained through the following representative example: S1UN refers to the family of SDFs with parent SDF $S_{p_1}(\kappa)$ defined in Eq. (33) and with Uniform marginal PDF defined in Table 1.

5.4. A note on computational demand

It should be noted that the GVRF methodology is computationally intensive. For each family, there are N stochastic fields $f_n(x)$; $n = 1, 2, \dots, N$ (defined through a parent SDF and a marginal PDF, e.g. S1UN), requiring N sets of Monte Carlo simulations (each set of simulations is necessary to determine one variance on the left-hand-side of Eq. (24)). These intensive computations are performed on an IBM Blue Gene supercomputer owned by Brookhaven National Laboratory using the IBM Fortran90 XL compiler. Each set of Monte Carlo simulations involves 102,400 deterministic runs that are distributed over 4,096 processors for a total of about 30 min of CPU time for all N sets of Monte Carlo simulations for a given family of N stochastic fields.

It is noted that the GVRF methodology is a brute-force procedure to explore the degree of approximation inherent in the GVRF methodology for certain (mostly statically indeterminate) structures. Once the GVRF is established for a specific structure, the variance of the effective flexibility can be computed for any stochastic field describing the uncertain system properties with minimal computational effort (a simple integration of the type shown in Eq. (21)). The initial upfront expense of the GVRF methodology becomes worthwhile if a large number of random fields are to be examined or especially if a sensitivity analysis is needed (there is no other way currently available to perform a full sensitivity analysis with respect to spectral properties). Furthermore, the task of simply determining a GVRF (without assessing the degree of approximation) is dramatically less computationally expensive than the full GVRF methodology since it can be done using only a small handful of random fields – this can be done when determining GVRFs for structures that are categorically similar to other structures where the GVRF methodology has been successfully performed.

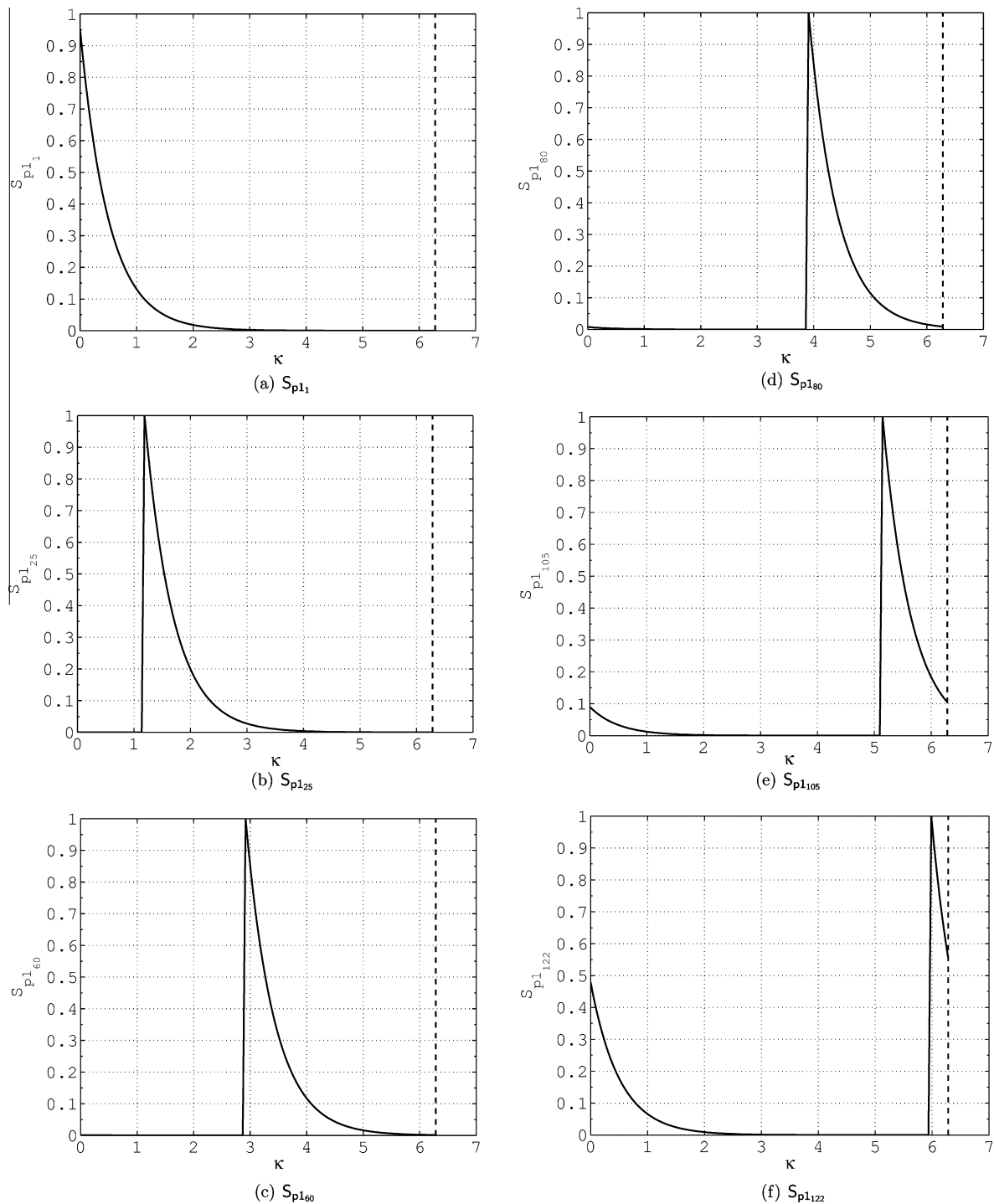


Fig. 3. Selected members of a family of SDFs of an underlying Gaussian field. The parent SDF for this family is shown in (a) and is defined in the first row of Eq. (33).

5.5. Example: GVRF of a fixed simply-supported beam

Consider the fixed-simply supported beam shown in Fig. 4 with $q = 1500 \text{ N/m}$, $L = 16 \text{ m}$, $E_0 I = 1.25 \times 10^7 \text{ N m}^2$, and $M = 7000 \text{ N m}$. Using the methodology described in Sections 5.1, 5.2, 5.3, GVRFs are computed for six pairs of parent SDFs and marginal PDFs: S1LN, S1TG, S1UN, S2LN, S2TG, S2UN. These computations are performed using Eq. (24). The variances in the left-hand-side vector in Eq. (24) are estimated by Monte Carlo simulations. For each deterministic analysis in these simulations, the effective flexibility is computed using Eq. (12).

The six resulting GVRFs are plotted in Fig. 5 where it is observed that they are very close to each other. The corresponding six pairs of parent SDFs and marginal PDFs are very different from each other. Studying Fig. 5 carefully, the following conclusions can be drawn:

1. The GVRFs are essentially independent of the spectral characteristics of the stochastic field $f(x)$ modeling the uncertain structural properties (i.e. the GVRFs are SDF-independent).
2. The GVRFs exhibit a slight dependence on the marginal PDF of $f(x)$. Specifically, the two GVRFs corresponding to the Lognormal

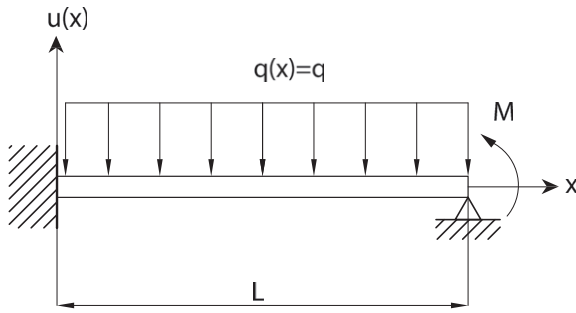


Fig. 4. Statically indeterminate fixed–simply supported beam analyzed.

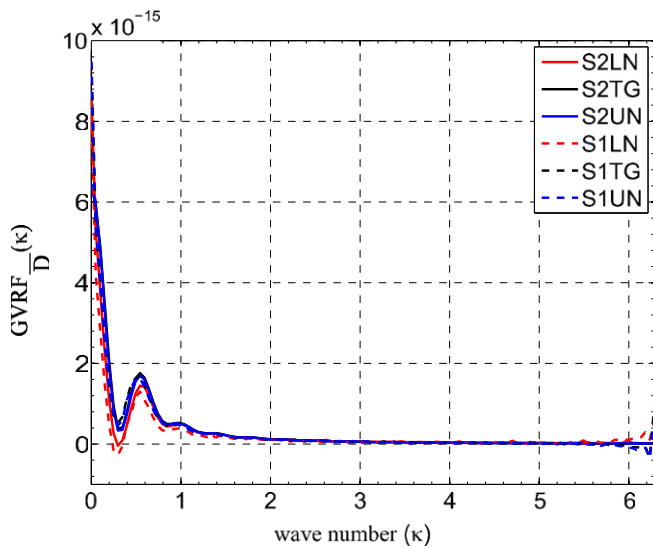


Fig. 5. GVRFs of effective flexibility for fixed–simply supported beam in Fig. 4.

marginal PDF are very similar to each other, but have some slight discrepancy from the GVRFs corresponding to the Uniform and truncated Gaussian distributions (the Lognormal PDF is significantly different from the Uniform and truncated Gaussian PDFs because it is skewed and has a tail that extends to $+\infty$).

The above conclusions are fortuitous because it is much easier to measure distributional characteristics (marginal PDF) than it is to measure spatial variation of material properties (SDF).

6. Validation of GVRFs

The validity of the GVRFs is tested by computing the coefficient of variation (COV) of the effective flexibility by brute-force Monte Carlo simulation for a different stochastic field model than the six ones used to determine the GVRFs, and comparing this value to the predicted COVs determined by the six GVRFs. The random field model chosen is a translation field whose marginal PDF is the Lognormal described in Table 1 and with an SDF that is determined from its underlying Gaussian field having the following SDF

$$S(\kappa) = \frac{1}{1.0233678} \exp(-3\kappa^2), \quad (34)$$

where the coefficient $1/1.0233678$ restrains the underlying Gaussian field to the desired unit variance. The predicted COVs from the six GVRFs are determined by performing the following simple integrations

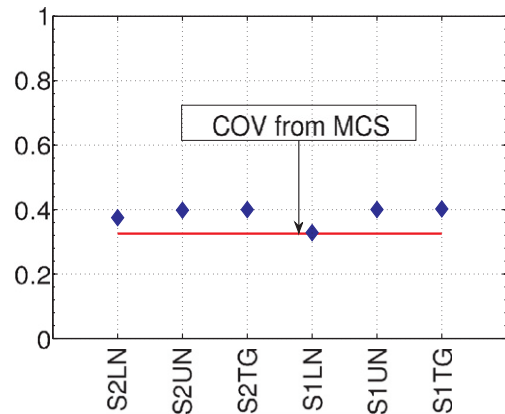


Fig. 6. Validation of GVRFs for fixed–simply supported beam. The red horizontal line represents the COV of the effective flexibility determined by Monte Carlo simulation. The blue diamonds represent the predicted COVs through integration of Eq. (35) and subsequent use of Eq. (36). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\text{Var}[\bar{D}] \approx \int_{-\kappa_{cl}}^{\kappa_{cu}} \text{GVRF}(\kappa) S_f(\kappa) d\kappa, \quad (35)$$

where $S_f(\kappa)$ is the SDF of the translated field, and then computing the COVs from

$$\text{COV} = \frac{\sqrt{\text{Var}[\bar{D}]}}{|\mathbb{E}[\bar{D}]|}. \quad (36)$$

Fig. 6 plots the results of this validation. The red horizontal line represents the COV of the effective flexibility computed through brute-force Monte Carlo simulation of the aforementioned random field model, and the blue diamonds represent the COVs predicted by performing the integration in Eq. (35) for each one of the six GVRFs computed. All six predictions (blue diamonds) are reasonably good, validating thus the computed GVRFs. The slight dependence of the GVRFs on the marginal PDF mentioned earlier is clearly observed in Fig. 6 where the two GVRFs corresponding to the two stochastic field models having a Lognormal marginal PDF provide better predictions than the other four random field models (corresponding to Uniform and truncated Gaussian marginal PDFs).

7. Conclusions

The main finding of this work is that the VRF concept is applicable for the effective flexibility of statically indeterminate structures via the GVRF methodology. GVRFs for the effective flexibility have been established and computed for a linear statically indeterminate beam. Although there are slight discrepancies amongst the six GVRFs computed (partially due to numerical and estimation errors), these GVRFs have been determined using six significantly different random field models. This result suggests that for this class of structures (i.e. statically indeterminate beams), the dependence of the GVRFs on the random field is very minor. Specifically, the GVRFs appear to be essentially SDF-independent and exhibit a slight dependence on the marginal PDF of the random field. This result is quite encouraging as regards to the application of the GVRF methodology to broad classes of structural systems that comprise beams and columns; and the full potential of the method can be realized if it is developed for stochastic effective properties of continuum bodies. The framework for an extension to continuum effective properties is already published [1] and the major obstacles appear to be computational in nature, involving the conditioning of Eq. (24), for example. Methods are available for surmounting these challenges such as efficient organization of the

SDFs in Eq. (24) and the use of inversion techniques well suited to poorly conditioned matrices. Thus, the key results of this paper represent an important stage in the development of a robust VRF theory for effective properties of continuum bodies.

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