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A Monte Carlo simulation model for stationary non-Gaussian processes

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Abstract

A class of stationary non-Gaussian processes, referred to as the class of mixtures of translation processes, is defined by their finite dimensional distributions consisting of mixtures of finite dimensional distributions of translation processes. The class of mixtures of translation processes includes translation processes and is useful for both Monte Carlo simulation and analytical studies. As for translation processes, the mixture of translation processes can have a wide range of marginal distributions and correlation functions. Moreover, these processes can match a broader range of second order correlation functions than translation processes. The paper also develops an algorithm for generating samples of any non-Gaussian process in the class of mixtures of translation processes. The algorithm is based on the sampling representation theorem for stochastic processes and properties of the conditional distributions. Examples are presented to illustrate the proposed Monte Carlo algorithm and compare features of translation processes and mixture of translation processes.

Keywords: Monte Carlo simulation; Non-Gaussian processes; Sampling theorem; Stochastic processes; Translation processes

1. Introduction

Current models for non-Gaussian processes can be divided in three classes: (1) memoryless transformations of $\mathbb{R}^d$-valued stationary Gaussian processes referred to as translation processes, (2) conditional Gaussian processes, for example, Gaussian processes with randomized spectral density [6] (3) diffusion and filtered Poisson processes, which represent states of non-linear and linear filters driven by Gaussian and Poisson white noise input. Conceptual simplicity and the ability to match any marginal distribution and a broad range of correlation functions are the main features of translation processes. A limitation of these models is their inability to capture higher order correlation functions [4]. Conditional Gaussian processes have useful properties in some applications [6]. Diffusion processes are difficult to calibrate to a specified marginal distribution and correlation function except for the case of the exponential correlation function [1,5]. Filtered Poisson processes can match any correlation function but cannot be calibrated to an arbitrary marginal distribution [5].

The objectives of this paper are to (1) define a class of stationary non-Gaussian processes X with continuous samples and finite second moment, which can match a broad class of finite dimensional distributions, and (2) develop a Monte Carlo simulation algorithm for generating samples of this process. A first difficulty in achieving these objectives is the limited availability of non-Gaussian multivariate distributions satisfying Kolmogorov's consistency and symmetry conditions. It is shown that a mixture of distributions derived from translation processes satisfies the Kolmogorov conditions. A second difficulty is numerical in nature. There are no efficient numerical algorithms for generating samples of non-Gaussian processes specified by their finite dimensional distributions. An algorithm is proposed for generating samples of the proposed non-Gaussian process. The algorithm is based on the sampling theorem for stochastic processes and properties of conditional distributions.

Two numerical examples are presented. The first example evaluates the feasibility and the accuracy of the proposed Monte Carlo simulation algorithm. The second example shows that, in addition to matching a wide range of marginal distributions and correlation functions, the proposed model can also represent a broader class of second order correlation functions than the translation models.

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2. Mixtures of distributions

There are very few multivariate distributions that can be used to define stochastic processes [10]. This section develops a class of multivariate distributions satisfying the consistency and symmetry conditions so that it can be used to define stochastic processes. These distributions are mixtures of finite dimensional distributions of translation processes.

Let $\{F_k(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^d\}$, $k=1,\ldots,m$, be a family of multivariate distributions. Denote by f_k the density of F_k. The mixtures of these distributions and densities are

$$F(\mathbf{x})=\sum_{k=1}^{m} p_k F_k(\mathbf{x}) \tag{1}$$

and

$$f(\mathbf{x})=\sum_{k=1}^{m} p_k f_k(\mathbf{x}), \tag{2}$$

respectively, where $p_k\geq 0$ and $\sum_{k=1}^m p_k=1$. The mixtures in Eqs. (1) and (2) with $m>1$ are said to be non-degenerate if the probabilities p_k satisfy the condition $p_k<1$ for all $k=1,\ldots,m$. The moments of the mixture in Eqs. (1) and (2) relate to the moments of its constituents by

$$\mu(q_1,\ldots,q_d)=\int_{\mathbb{R}^d}\prod_{i=1}^{d} x_i^{q_i}\,\mathrm{d}F(\mathbf{x})$$

$$=\sum_{k=1}^{m} p_k\int_{\mathbb{R}^d}\prod_{i=1}^{d} x_i^{q_i}\,\mathrm{d}F_k(\mathbf{x})=\sum_{k=1}^{m} p_k\mu_k(q_1,\ldots,q_d), \tag{3}$$

where $q_i\geq 0$ are integers, $\mu(q_1,\ldots,q_d)$ is a moment of order $q=\sum_{i=1}^d q_i$ of F, and $\mu_k(q_1,\ldots,q_d)$ denotes the corresponding moment of F_k. If distributions F_k, $k=1,\ldots,m$, have finite moments of order q, the mixture of distributions F in Eq. (1) has the same property.

3. The class of mixtures of translation processes

Consider a collection of independent translation processes,

$$X_k(t)=G_k^{-1}\circ\Phi(Y_k(t))=h_k(Y_k(t)),\qquad k=1,\ldots,m, \tag{4}$$

where Y_k are stationary Gaussian processes with mean zero, variance one, correlation function $\rho_k(\tau)=E[Y_k(t)Y_k(t+\tau)]$, and spectral density $s_k(\nu)$, Φ denotes the distribution of a standard Gaussian variable, and G_k is a continuous distribution with density g_k, mean zero, and variance one. The correlation function of X_k can be calculated from

$$\xi_k(\tau)=E[X_k(t)X_k(t+\tau)]$$

$$=\int_{\mathbb{R}^2} h_k(\alpha)h_k(\beta)\phi(\alpha,\beta;\rho_k(\tau))\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{5}$$

where $\phi(\cdot,\cdot,\rho_k(\tau))$ is the joint density of a standard bivariate Gaussian vector with correlation coefficient $\rho_k(\tau)$ [4].

Let $t_1,\ldots,t_d$ be arbitrary times. The joint distribution and density of the vector $(X_k(t_1),\ldots,X_k(t_d))$ are

$$F_k(x_1,\ldots,x_d;t_1,\ldots,t_d)=P\left(\bigcap_{i=1}^{d}\{Y(t_i)\leq y_i\}\right)$$

$$=\Phi(y_1,\ldots,y_d;\boldsymbol{\rho}_k) \tag{6}$$

and

$$f_k(x_1,\ldots,x_d;t_1,\ldots,t_d)$$

$$=[(2\pi)^d\det(\boldsymbol{\rho}_k)]^{-1/2}\prod_{i=1}^{d}\frac{g_k(x_i)}{\phi(y_i)}\exp\left(-\frac{1}{2}\mathbf{y}^{\mathrm{T}}\boldsymbol{\rho}_k\mathbf{y}\right), \tag{7}$$

where $\boldsymbol{\rho}_k=\{\rho_k(t_i-t_j)\}$, $i,j=1,\ldots,d$, is the covariance matrix of $(Y(t_1),\ldots,Y(t_d))$, $\Phi(\cdot,\ldots,\cdot;\boldsymbol{\rho}_k)$ is the joint distribution of this vector, and $y_i=\Phi^{-1}\circ G_k(x_i)$, $i=1,\ldots,d$. The functions in Eqs. (6) and (7) are referred to as multivariate translation distribution and density functions, respectively. Let

$$F(x_1,\ldots,x_d;t_1,\ldots,t_d)=\sum_{k=1}^{m} p_k\Phi(y_1,\ldots,y_d;\boldsymbol{\rho}_k) \tag{8}$$

and

$$f(x_1,\ldots,x_d;t_1,\ldots,t_d)=\sum_{k=1}^{m} p_k[(2\pi)^d\det(\boldsymbol{\rho}_k)]^{-1/2}\prod_{i=1}^{d}\frac{g_k(x_i)}{\phi(y_i)}$$

$$\times\exp\left(-\frac{1}{2}\mathbf{y}^{\mathrm{T}}\boldsymbol{\rho}_k\mathbf{y}\right), \tag{9}$$

be the mixtures of distributions and densities in Eqs. (1) and (2), respectively, with F_k and f_k in Eqs. (6) and (7). The multivariate distribution and density in Eqs. (8) and (9) are referred to as mixtures of translation distribution and density functions, respectively.

Let $\mathscr{X}$ denote the collection of stochastic processes defined by the finite dimensional distributions and densities in Eqs. (8) and (9). A member X of $\mathscr{X}$ is called a *mixture of translation processes*. Since the finite dimensional distributions F_k, $k=1,\ldots,m$, define translation processes, they satisfy the symmetry and consistency conditions [2]. Hence, the finite dimensional distributions in Eq. (8) satisfy the same conditions so that the class of processes $\mathscr{X}$ is well defined. The processes in $\mathscr{X}$ and their distributions have the following properties.

(1) *Translation distributions are degenerate versions of mixtures of translation distributions.* Take $m=2$ in Eqs. (8)

and (9). Suppose that f in Eq. (8) is not degenerate and that the translation densities f_k of order two satisfy the condition $f_k(x_1,x_2)=f_k^{(1)}(x_1)f_k^{(1)}(x_2)$, where $f_k^{(1)}$ denote probability density functions, $k=1,\ 2$. Hence, the mixture $f=pf_1+(1-p)f_2,\ p\in(0,1)$, in Eq. (9) has uncorrelated coordinates. If f is a translation density, then

$$pf_1(x_1,x_2)+(1-p)f_2(x_1,x_2)$$
$$=(pf_1^{(1)}(x_1)+(1-p)f_2^{(1)}(x_1))(pf_1^{(1)}(x_2)+(1-p)f_2^{(1)}(x_2))$$

since uncorrelated translation variables are independent. The above equality gives

$$(f_1^{(1)}(x_1)-f_2^{(1)}(x_1))(f_1^{(1)}(x_2)-f_2^{(1)}(x_2))=0$$

for all $(x_1,x_2)\in\mathbb{R}^2$. This implies that $f_1^{(1)}$ coincides with $f_2^{(1)}$ so that the mixture is degenerate in contradiction with the initial assumption.

(2) *The multivariate distribution and density functions in Eqs.* (8) *and* (9) *define a stationary stochastic process X*. It has already been shown that the class of processes $\mathscr{X}$ is well defined. The members of $\mathscr{X}$ are stationary processes since the distributions $F_k,\ k=1,\ldots,m$, are invariant to a time shift.

(3) *The moments of any order of X are*

$$\mu(q_1,\ldots,q_d;t_1,\ldots,t_d)=E\left[\prod_{i=1}^{d}X(t_i)^{q_i}\right]$$
$$=\int_{\mathbb{R}^d}\prod_{i=1}^{d}x_i^{q_i}\,\mathrm{d}F(\mathbf{x})=\sum_{k=1}^{m}p_k\int_{\mathbb{R}^d}\prod_{i=1}^{d}x_i^{q_i}\,\mathrm{d}F_k(\mathbf{x})$$
$$=\sum_{k=1}^{m}p_kE\left[\prod_{i=1}^{d}X_k(t_i)^{q_i}\right]$$
$$=\sum_{k=1}^{m}p_k\mu_k(q_1,\ldots,q_d;t_1,\ldots,t_d),\qquad(10)$$

where $\mu_k(q_1,\ldots,q_d;t_1,\ldots,t_d)=E[\prod_{i=1}^{d}X_k(t_i)^{q_i}]$ *is the moment of process* $X_k(t)$. The definition of the moments of a random vector and Eqs. (8) and (9) yield Eq. (10). Because the translation processes X_k are stationary, the moments $E[\prod_{i=1}^{d}X_k(t_i)^{q_i}]$ are invariant to a time shift so that the moments $\mu(q_1,\ldots,q_d;t_1,\ldots,t_d)$ depend only on the time lags $(t_2-t_1,\ldots,t_d-t_1)$ rather than the times $(t_1,\ldots,t_d)$. For $d=2$ and $q_1=q_2=1$, Eq. (10) gives

$$\mu(1,1;t_1,t_2)=\sum_{k=1}^{m}p_k\mu_k(1,1;t_2-t_1),\qquad(11)$$

(4) *The marginal distribution of X is*

$$F^{(1)}(x)=\sum_{k=1}^{m}p_kF_k^{(1)}(x)=\sum_{k=1}^{m}p_kG_k(x).\qquad(12)$$

The first equality holds since the distributions F_k satisfy the consistency condition. The second equality is just a notation (Eq. (4)).

(5) *The finite dimensional density of X in Eq.* (9) *degenerates into a one-dimensional distribution with probability mass on a line equally inclined relative to the coordinates of* $\mathbb{R}^d$ *as* $(t_1,\ldots,t_d)\to t$. This property must be satisfied by the finite dimensional distributions of any process X since the random variables $X(t_1),\ldots,X(t_d)$ coincide in the limit as $(t_1,\ldots,t_d)\to t$. In particular, it is satisfied by the translation distributions. Eq. (8) implies that the mixture of translation distributions has the same property.

(6) *If the processes in the mixture are type A ergodic, then X is ergodic of type A*. For example, suppose that all processes X_k have mean zero and are ergodic in the mean, that is, the estimator

$$X_{k,\tau}=\frac{1}{2\tau}\int_{-\tau}^{\tau}X_k(s)\mathrm{d}s$$

of the mean of X_k has the properties $E[X_{k,\tau}]=E[X_k(t)]=0$ and $\mathrm{Var}[X_{k,\tau}]\to 0$ as $\tau\to\infty$. The corresponding estimator,

$$X_\tau=\frac{1}{2\tau}\int_{-\tau}^{\tau}X(s)\mathrm{d}s,$$

of the mean of X is unbiased and its variance approaches zero as τ increases indefinitely since

$$E[X_\tau]=\frac{1}{2\tau}\int_{-\tau}^{\tau}E[X(s)]\mathrm{d}s=\frac{1}{2\tau}\int_{-\tau}^{\tau}\left(\int_{\mathbb{R}}u\,\mathrm{d}F^{(1)}(u)\right)$$
$$=\sum_{k=1}^{m}p_kE[X_{k,\tau}]=0$$
$$\mathrm{Var}[X_\tau]=\frac{1}{4\tau^2}\int_{-\tau}^{\tau}\int_{-\tau}^{\tau}E[X(s)X(t)]\mathrm{d}s\,\mathrm{d}t$$
$$=\frac{1}{4\tau^2}\int_{-\tau}^{\tau}\int_{-\tau}^{\tau}\left(\int_{\mathbb{R}^2}uv\,\mathrm{d}F(u,v;s,t)\right)\mathrm{d}s\,\mathrm{d}t$$
$$=\sum_{k=1}^{m}p_k\,\mathrm{Var}[X_{k,\tau}]\to 0$$

$\tau\to\infty$.

Similar considerations can be used to prove other types of ergodicity.

(7) *The process X is completely defined by the probabilities* $p_1,\ldots,p_{m-1}$, *the marginal distributions* $G_1,\ldots,G_m$, *and the correlation functions* $\rho_1,\ldots,\rho_m$. The defining parameters can be estimated from a record of the target process following these steps. First, the method in Ref. [8] can be applied to select a mixture $\sum_{k=1}^{m}p_kG_k$ for the marginal distribution of X based on estimates of the marginal moments of the record. This step defines the marginal distributions G_k and their weights p_k. Second, estimates $\hat{\xi}$ and $\hat{\zeta}$ of the first and second order correlation functions $\xi=E[X(t)X(t+\tau)]$ and $\zeta=E[X(t)X(t+\tau)X(t+\sigma)]$ of the record can be used to select optimal functional forms for the correlation functions ρ_k such that the difference between the estimates $\hat{\xi}$ and $\hat{\zeta}$ and the first and

second order correlation functions ξ and ζ of X be minimized in some sense.

4. Time domain representation of processes in $\mathscr{X}$

Let $\{x(t),\ t \in \mathbb{R}\}$ be a deterministic function whose Fourier transform is zero outside a bounded frequency range $(-\nu_0, \nu_0)$, $0 < \nu_0 < \infty$. The sampling theorem gives the representation [3]

$$x(t) = \lim_{m\to\infty} \sum_{k=-m}^{m} x(k\tau)\alpha_k(t), \tag{13}$$

where $\tau = \pi/\nu_0$, times $k\tau$ are called nodes, and

$$\alpha_k(t) = \frac{\sin(\pi(t/\tau - k))}{\pi(t/\tau - k)}. \tag{14}$$

Hence, x is completely defined by its values at $k\tau$, $k = 0, \pm 1, \pm 2, \ldots$, that is, it is sufficient to sample x at a rate equal to half of its shortest period. The sampling theorem has been used to represent stationary Gaussian processes and develop an algorithm for generating samples of these processes [3,4]. These developments are now extended to the class of stationary non-Gaussian processes defined in Section 3.

Let $X \in \mathscr{X}$ be a stationary non-Gaussian process with finite dimensional distributions in Eq. (8). If the spectral density of X is zero outside a frequency band $(-\nu_0, \nu_0)$, then almost all samples of this process can be represented by harmonics with frequencies in the range $(-\nu_0, \nu_0)$ so that Eqs. (13) and (14) yield

$$X(t,\omega) = \lim_{m\to\infty} \sum_{k=-m}^{m} X(k\tau,\omega)\alpha_k(t) \tag{15}$$

for almost all ωs, where ω is an element of the sample space. If the spectral density functions of the translation processes X_k in the definition of $X \in \mathscr{X}$ are zero outside the frequency bands $(-\nu_{k,0}, \nu_{k,0})$, $k = 1, \ldots, m$, then the spectral density of X is zero in $(-\max_{1\le k\le m}\{\nu_{k,0}\}, \max_{1\le k\le m}\{\nu_{k,0}\})^c$. Hence, the sampling theorem can be applied to represent almost all samples of X.

The exact representation of X in Eq. (15) cannot be used in calculations since it involves an infinite number of terms and random variables. Consider the approximation

$$X_n(t,\omega) = \sum_{k=n_t-n}^{n_t+n+1} X(k\tau,\omega)\alpha_k(t) \tag{16}$$

of X, where $n_t = [t/\tau]$ is the largest integer smaller than t/τ and the integer $n > 0$ gives the number of nodes right and left of the cell $[n_t\tau, (n_t+1)\tau]$ containing the current time

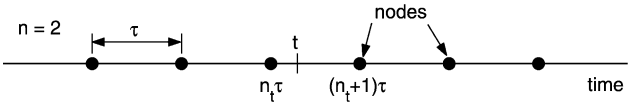

Fig. 1. Approximate representation of X.

(Fig. 1). This approximation depends on the values of X at $2(n+1)$ nodes, and has the property $X_n(k\tau,\omega) = X(k\tau,\omega)$ for $k = n_t - n, \ldots, n_t + n + 1$, since $\alpha_k(l\tau) = 1$ for $k = l$ and $\alpha_k(l\tau) = 0$ for $k \neq l$. A Monte Carlo simulation algorithm for generating samples of X_n is presented in Section 5.

The accuracy of the approximate representation X_n depends on the size n of the window used in the definition of X_n (Eq. (16)). Let

$$U_n(t) = X(t) - X_n(t) \tag{17}$$

be the approximation error at a time $t \in [n_t\tau, (n_t+1)\tau]$. Several measures can be used to quantify this error. For example, the probabilities $P(\max_{t\in[n_t\tau,(n_t+1)\tau]}|U_n(t)| > \epsilon)$ or $P(|U_n((n_t+1/2)\tau)| > \epsilon)$, where $\epsilon > 0$ is a small number. A heuristic justification for evaluating the error at the cell midpoint is that X_n coincides with X at the nodes so that the error is likely to increase with the distance from the nodes. The calculation of these probabilities can be very difficult. A simpler measure, the mean square error $e = E[U_n((n_t + 1/2)\tau)^2]$, is considered in the following example.

Example 1. Let Y be a stationary Gaussian process with mean zero, unit variance, one-sided spectral density of intensity $1/\nu_0$ in the frequency band $(0, \nu_0)$ and zero outside it, and correlation function $\rho(s) = \sin(\nu_0 t)/(\nu_0 t)$. The translation process $X(t) = Y(t)^3/\sqrt{15}$ has mean zero, unit variance and covariance function $\xi(t) = \rho(t)(3 + 2\rho(t)^2)/5$ [4]. In this case it is possible to obtain an explicit formula for the mean square error e.

The Gaussian vector

$$\mathbf{Y} = [Y((n_t - n)\tau), \ldots, Y(n_t\tau), Y((n_t + 1/2)\tau), Y((n_t + 1)\tau), \ldots, Y((n_t + n + 1)\tau)]$$

has dimension $2n + 3$, mean zero, and covariance matrix $\boldsymbol{\rho} = \{\rho((k-l)\tau)\}$, $k, l = n_t - n, \ldots, n_t, n_t + 1/2, n_t + 1 \ldots, n_t + n + 1$. The corresponding vector

$$\mathbf{X} = [X((n_t - n)\tau), \ldots, X(n_t\tau), X((n_t + 1/2)\tau), X((n_t + 1)\tau), \ldots, X((n_t + n + 1)\tau)]$$

has also mean zero and covariance matrix $\boldsymbol{\xi} = \{\xi((k-l)\tau)\}$, $k, l = n_t - n, \ldots, n_t + 1/2, \ldots, n_t + n + 1$. The mean square error is

$$e = E\left[\left(X((n_t + 1/2)\tau) - \sum_{k=n_t-n}^{n_t+n+1} X(k\tau)\alpha_k((n_t + 1/2)\tau)\right)^2\right] = \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\xi}\boldsymbol{\beta}, \tag{18}$$

where $\boldsymbol{\beta}$ is a column vector with dimension $2n + 3$ and coordinates $(\beta_1 = -\alpha_{n_t-n}((n_t + 1/2)\tau), \ldots, \beta_{n+2} = 1, \ldots, \beta_{2n+3} = -\alpha_{n_t+n+1}((n_t + 1/2))$. Fig. 2 shows the variation of the mean square error in Eq. (18) with the window

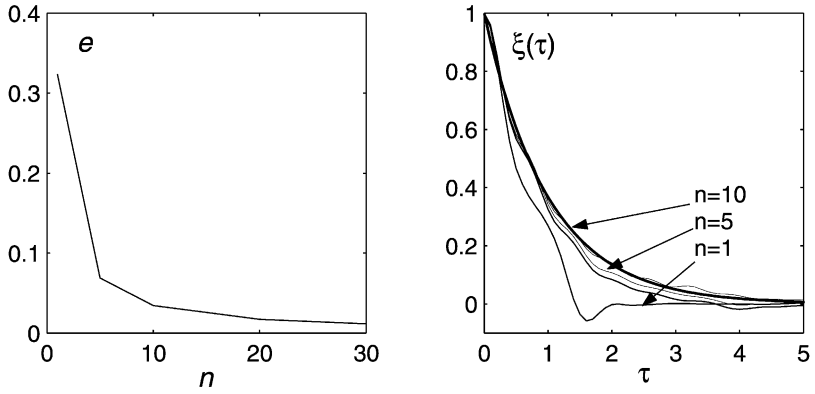

Fig. 2. Mean square error of the approximation X_n and correlation functions of X_n for several values of n.

size n for a band limited Gaussian white noise process with unit variance and bounding frequency $\nu_b = 5$ and a nodal spacing $\tau = \pi/(3\nu_b)$. As expected the approximate representation improves as n increases. The figure also shows the exact correlation function ξ and its approximations for $n = 1$, 5, and 10. The correlation function of X_n with $n = 10$ nearly coincides with the correlation function ξ of X.

There are two sources of error in the approximation X_n. First, the spectral density $s(\nu)$ is replaced by the spectral density $\tilde{s}(\nu) = 1_{(-\nu_0,\nu_0)}(\nu)s(\nu)$ for some $\nu_0 > 0$. Since the processes in $\mathscr{X}$ have finite variance, the integral $\int_{-\infty}^{\infty} s(\nu)\mathrm{d}\nu$ is finite so that $\int_{(-\nu_0,\nu_0)^c} s(\nu)\mathrm{d}\nu \to 0$ as $\nu_0 \to \infty$. Hence, any process $X \in \mathscr{X}$ can be approximated by a process with a bounded frequency range, that is, a process with spectral density $\tilde{s}(\nu) = 1_{(-\nu_0,\nu_0)}(\nu)$, $\nu \in (-\infty, \infty)$, provided that ν_0 is sufficiently large. Second, the algorithm for generating samples of X outlined in Section 5 uses only a finite number of values of this process determined by the value of the parameter n in Eq. (16). Extensive calculations show that accurate representations of X result for $n \simeq 3$–5 [7].

5. Monte Carlo simulation algorithm

Suppose a sample $X_n(t,\omega)$ of X_n has been generated for $t \le (n_t + 1)\tau$. The objective is to extend this sample into the next cell, that is, the time interval $[(n_t + 1)\tau, (n_t + 2)\tau]$. This extension requires a sample of the process at the node $n_t + n + 2$, that is, a sample of the conditional random variable

$$X((n_t + n + 2)\tau)|(X((n_t + n + 1)\tau) = X((n_t + n + 1)\tau, \omega), X((n_t + n)\tau) = X((n_t + n)\tau, \omega), \ldots). \tag{19}$$

This exact formulation is not practical because it requires conditioning on the entire past history, that is, a vector of increasing size as time progresses. Moreover, the contribution of values of X at nodes far away from the cell containing the current time is likely to be negligible. It is proposed to approximate the conditional random variable in Eq. (19) by [3,7].

$$\begin{aligned}\hat{X}((n_t + n + 2)\tau) &= X((n_t + n + 2)\tau)|X((n_t + n + 1)\tau) \\ &= X((n_t + n + 1)\tau, \omega), \\ X((n_t + n)\tau) &= X((n_t + n)\tau, \omega), \ldots, X((n_t - n + 1)\tau) \\ &= X((n_t - n)\tau, \omega)),\end{aligned} \tag{20}$$

that is, by the random variable $X((n_t + n + 2)\tau)$ conditioned on the values of X at the past $2(n + 1)$ nodes. Hence, the past history is represented in this approximation by a vector with the same dimension at all times.

The generation of samples of $\hat{X}((n_t + n + 2)\tau)$ is very simple for stationary Gaussian processes since the second moment properties of the vector $(X((n_t + n + 2)\tau), \ldots, X((n_t - n)\tau)$ are time invariant and define completely its probability law. The generation of samples of $\hat{X}((n_t + n + 2)\tau)$ is much more complicated if X is a stationary non-Gaussian process.

Let $f^{(2n+2)}$ and $f^{(2n+3)}$ be the joint density functions of $(X((n_t - n)\tau), \ldots, X((n_t + n + 1)\tau))$ and $(X((n_t - n)\tau), \ldots, X((n_t + n + 1)\tau), X((n_t + n + 2)\tau))$, respectively. Let $(X((n_t - n)\tau, \omega) = \zeta_1, \ldots, X((n_t + n + 1)\tau, \omega) = \zeta_{2n+2})$ be a sample of the first vector. The density of the conditional vector in Eq. (20) is

$$\hat{f}(x|\boldsymbol{\zeta}) = \frac{f^{(2n+3)}(\boldsymbol{\zeta}, x)}{f^{(2n+2)}(\boldsymbol{\zeta})}, \tag{21}$$

where $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{2n+2})$. There is no simple and efficient way to generate samples from the density $\hat{f}(x|\boldsymbol{\zeta})$ since the vector $\boldsymbol{\zeta}$ changes in time. The following algorithm has been used in this paper. Let u be a sample of a random variable uniformly distributed in $(0,1)$ and denote by (a,b) the range of $\hat{f}(\cdot|\boldsymbol{\zeta})$ used for numerical calculations. If $u \le 0.5$, integrate the conditional density from the left to find x such that $\int_a^x \hat{f}(\alpha|\boldsymbol{\zeta})\mathrm{d}\alpha = u$. If $u > 0.5$, integrate from the right to find x satisfying the condition $\int_x^b \hat{f}(\alpha|\boldsymbol{\zeta})\mathrm{d}\alpha = 1 - u$. The solution x is a sample of $\hat{f}(\cdot|\boldsymbol{\zeta})$. This algorithm has been used for Monte Carlo simulation. The following three steps can

be followed to produce a sample of the approximation X_n of X in a time interval $[0,\tilde{t}]$.

Step 1
Generate a sample ζ_1 of $X(0)$. Use this sample to generate a sample ζ_2 of $X(\tau)|(X(0)=\zeta_1)$. Then generate a sample ζ_2 of $X(2\tau)|(X(0)=\zeta_1, X(\tau)=\zeta_2)$. Continue this generation to obtain a vector $(\zeta_1,\ldots,\zeta_{2n+2})$. The generation of the samples $\zeta_2,\ldots,\zeta_{2n+2}$ is based on conditional densities of the type in Eq. (21).
Step 2
Use the conditional density in Eq. (21) to generate a sample of $X((n_t+n+2)\tau)$ given the values of X at the previous $2n+2$ nodes. This new value of X allows advancement of the simulation from cell $[n_t\tau,(n_t+1)\tau]$ to cell $[(n_t+1)\tau,(n_t+2)\tau]$. Repeat this step to produce a sample of X at all nodes in $[0,\tilde{t}]$.
Step 3
Calculate the corresponding sample of X_n from Eq. (16).

It has been shown that processes defined by finite dimensional distributions in Eq. (8) are not necessarily translation processes, provided that these distributions are not degenerate. The following examples demonstrate two features of the class of mixtures of translation processes defined in this paper. These processes exhibit intermittent behavior and can describe second order correlation functions that cannot be matched by translation processes. These feature of the mixtures of translation processes can be very useful in some applications. For example, if the available information on a time series consists of the marginal distribution and the first and second order correlation functions [9], translation processes can be inadequate.

Example 2. Let $m=2$ in Eqs. (6)–(9) and let $X_k(t)=F_U\circ\Phi(Y_k(t))$, $k=1,\ 2$, where Y_1 and Y_2 are stationary Gaussian processes with mean zero and covariance functions

$$\rho_1(\alpha)=\mathrm{e}^{-\lambda|\alpha|} \tag{22}$$

$$\rho_2(\alpha)=\frac{\sin((\nu_b-\nu_a\alpha/2)\cos((\nu_b+\nu_a\alpha/2)}{(\nu_b-\nu_a)\alpha/2},$$

$$0\le\nu_a<\nu_b.$$

The processes Y_1 and Y_2 are independent of each other. The distribution in the definition of the processes X_k is $F_U(x)=(1/2)(x+1)1_{[-1,1]}(x)+1_{[1,\infty)}(x)$, that is, the random variables $X_k(t)$ are uniformly distributed in $[-1,1]$ at each time $t\ge 0$. The function $1_{[a,b]}(x)$ is 1 for $x\in[a,b]$ and 0 otherwise.

Fig. 3 shows five samples of X with finite dimensional distributions in Eq. (8) corresponding to $p_1=0$, 1/4, 1/2,

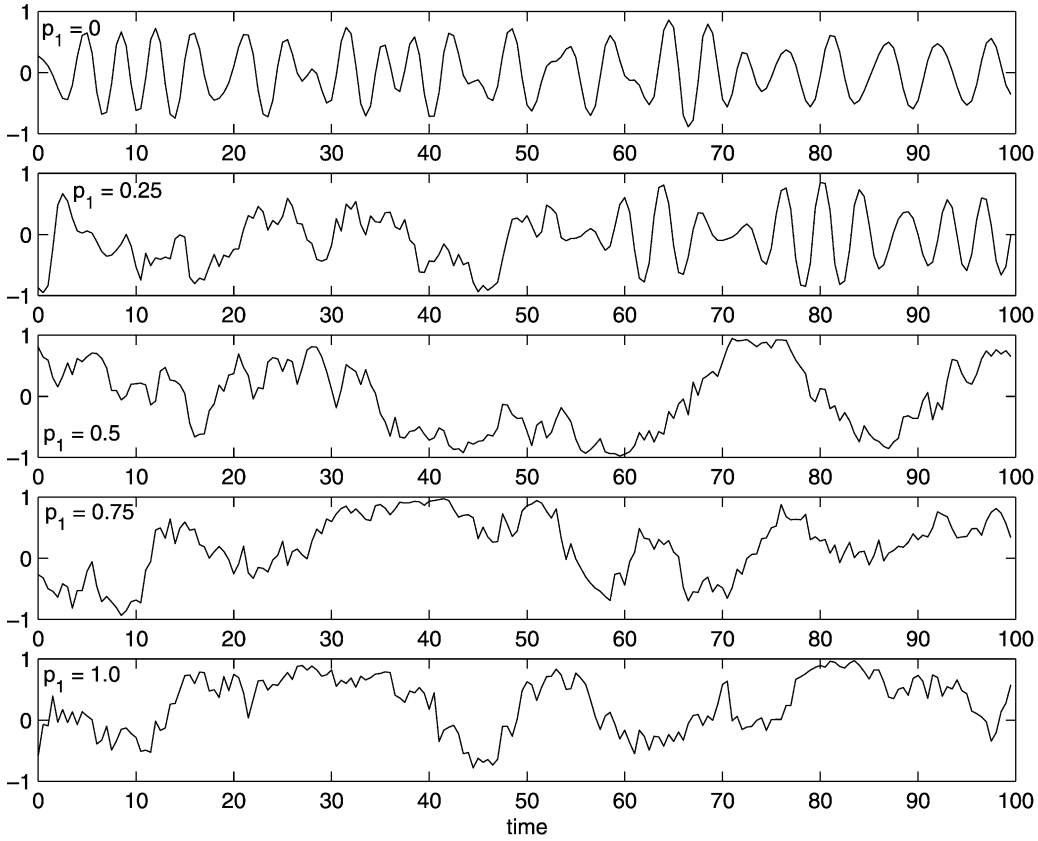

Fig. 3. Five samples of $X(t)$ corresponding to $p_1=0$, 1/4, 1/2, 3/4, and 1.

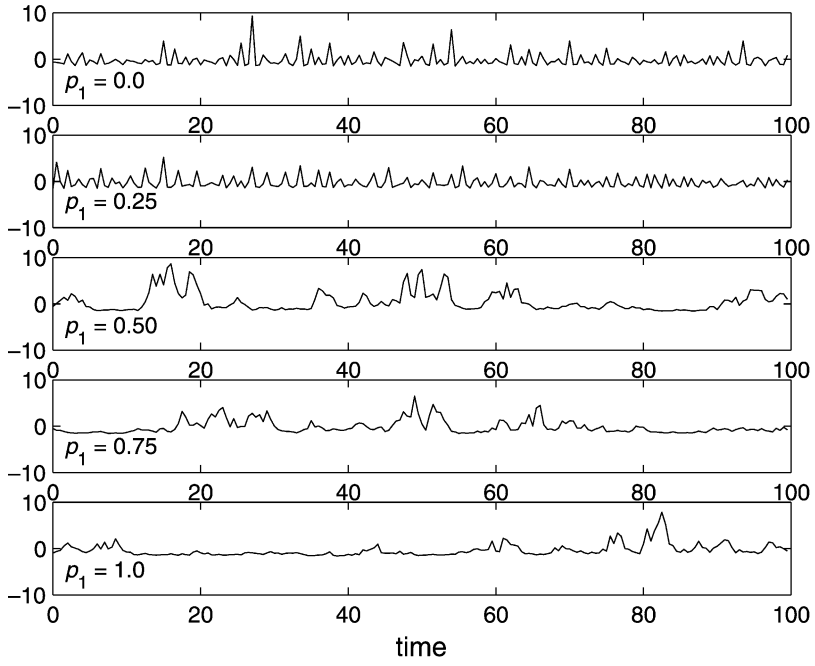

Fig. 4. Five samples of $X(t)$ corresponding to $p_1 = 0$, 1/4, 1/2, 3/4, and 1.

3/4, and 1. Numerical results are for $\lambda = 0.1$, $\nu_a = 1$, and $\nu_b = 2$. The samples have been generated by the Monte Carlo algorithm in this section using a nodal spacing $\tau = 0.25$ and a window size $n = 5$. The samples of X coincide with the samples of X_1 and X_2 for $p_1 = 0$ and $p_1 = 1$, respectively. For other values of p_1 the samples of X incorporate features of both X_1 and X_2. The sample of X for $p_1 = 1/4$ appears to be alternately dominated by the sample properties of X_1 and X_2.

The samples in Fig. 3 suggest that the processes in $\mathscr{X}$ can model intermittent behavior. Such behavior can be observed in some applications, for example, the wind speed process can alternate between two patterns of behavior corresponding to smooth and turbulent flow. The variation of soil properties with depth in geological deposits with randomly alternating soil layers characterized by random properties can also exhibit intermittent behavior. The sample in Fig. 3 have features that are consistent with property **6** in Section 4 defining the class of processes $\mathscr{X}$. For example, if the constituent translation processes X_k are ergodic in the marginal distribution and the correlation function, the corresponding process $X \in \mathscr{X}$ is also ergodic in the marginal distribution and correlation function. Hence, the values and the frequency content of each sample of X have to reflect the corresponding features of the constituent processes X_k, and these features have to be incorporated in the proportion p_k.

Example 3. Let $m = 2$ in Eqs. (6)–(9) and let

$$X_k(t) = \frac{e^{Y_k(t)} - e^{1/2}}{\sqrt{e^2 - e}}, \qquad k = 1, 2, \tag{23}$$

in Eq. (4), where Y_k are stationary Gaussian processes with mean zero and covariance functions in Eq. (22). The marginal distribution and density of the processes X_k are

$$F(x)\Phi(\log(x\sqrt{e-1}+1)+1/2)) \tag{24}$$

$$f(x) = \frac{\sqrt{e-1}}{x\sqrt{e-1}+1}\,\phi(\log(x\sqrt{e-1}+1)+1/2)$$

for $x > -e/\sqrt{e^2-e}$. Let X be a non-Gaussian process in $\mathscr{X}$ with the finite dimensional distributions in Eqs. (6)–(9), $m = 2$, and processes X_k, $k = 1$, 2, with the marginal distribution and density in Eq. (24).

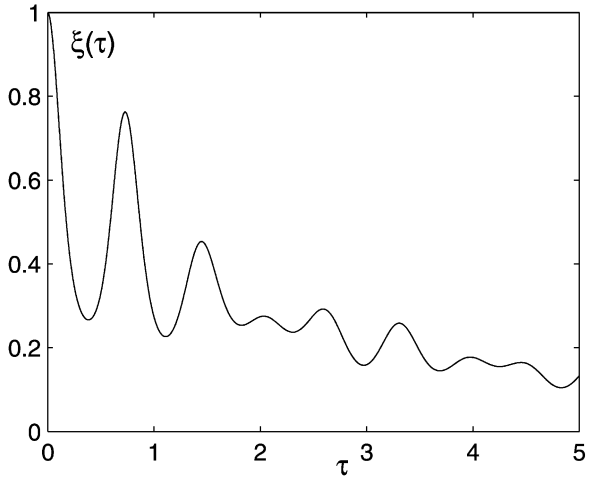

Fig. 5. First order correlation function ξ of X for $p_1 = 1/2$.

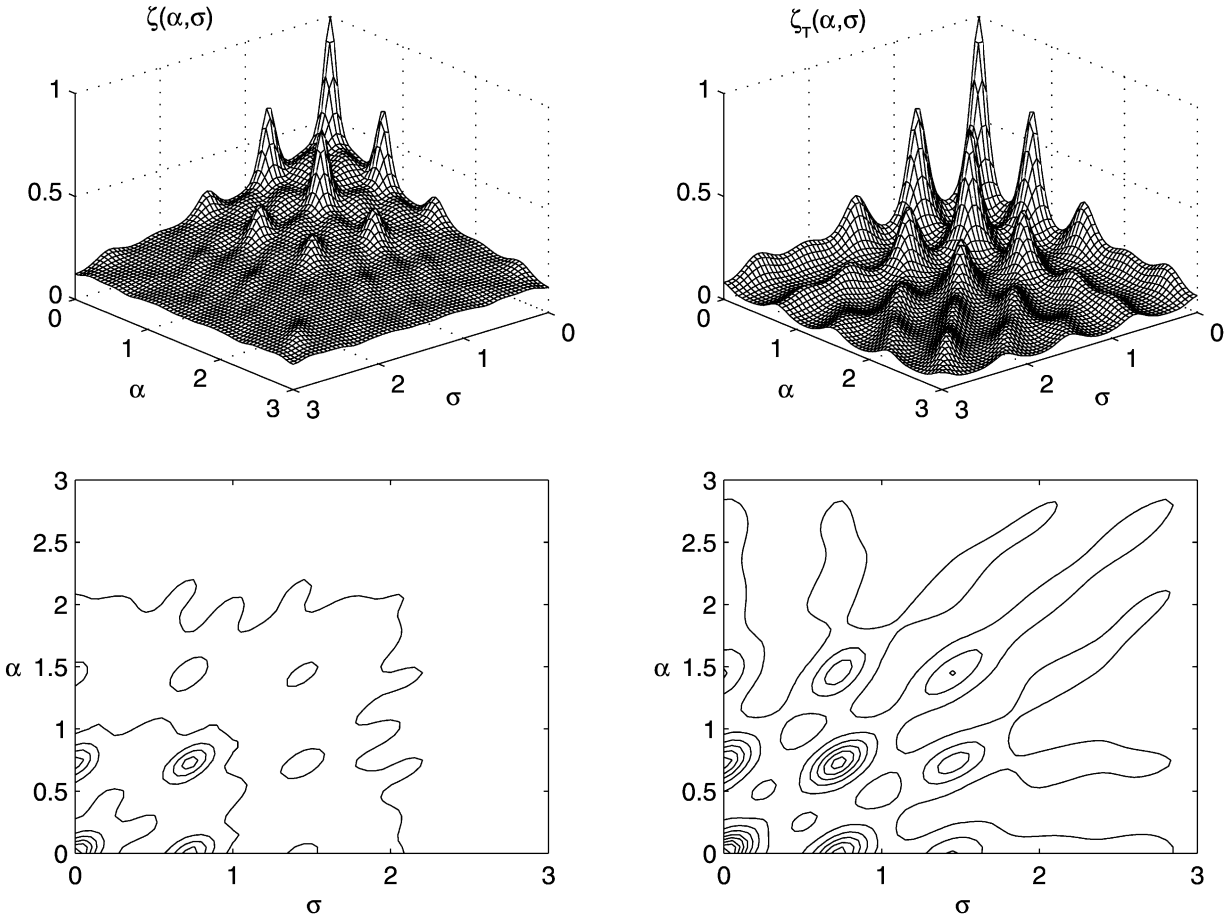

Fig. 6. Second order correlation functions ζ and ζ_T of X and X_T, respectively.

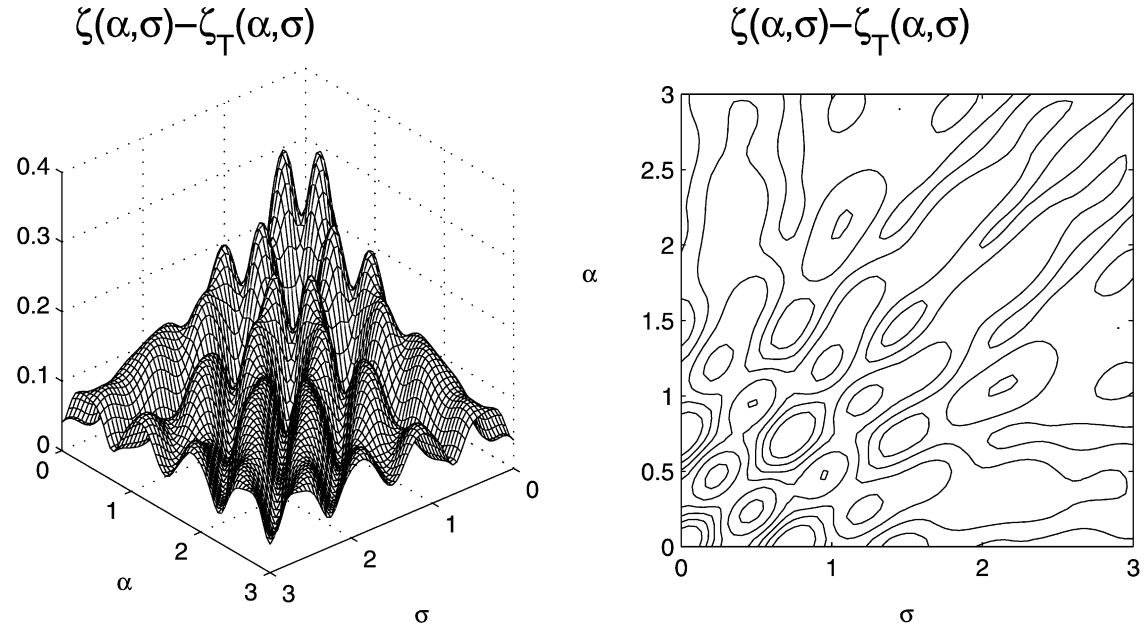

Fig. 7. Difference $\zeta - \zeta_T$ of the second order correlation functions of X and X_T.

The first and second order correlation functions of the processes X_k are

$$\xi_k(\alpha) = E[X_k(t)X_k(t+\alpha)]\frac{e^{\rho_k(\alpha)} - 1}{e - 1} \tag{25}$$

$$\zeta_k(\alpha,\sigma) = E[X_k(t)X_k(t+\alpha)X_k(t+\sigma)] = \frac{e^{\rho_k(\alpha)+\rho_k(\sigma)+\rho_k(\alpha-\sigma)}}{(e-1)^{3/2}},$$

where ρ_k, $k = 1,2$, are in Eq. (22).

Fig. 4 shows five samples of X for $\lambda = 1/5$, $\nu_a = 7$, and $\nu_b = 10$ corresponding to $p_1 = 0$, 1/4, 1/2, 3/4, and 1. The samples have been generated by the Monte Carlo algorithm in this section using a nodal spacing $\tau = 0.1$ and a window size $n = 5$. The samples illustrate the dependence of the correlation structure of X on the correlation functions of the constituent processes X_k and the weights p_k of these processes in the definition of X. Fig. 5 shows the first order correlation function of X for $p_1 = 1/2$. Consider also a translation process X_T with the marginal distribution F in Eq. (24) and the covariance function in Fig. 5. Fig. 6 shows three dimensional views and contour lines of the second order correlation functions ζ and ζ_T of X and of the translation process X_T. Fig. 7 shows a three-dimensional view and contour lines of the difference $\zeta - \zeta_T$ between the second order correlation functions of X and X_T. Figs. 6 and 7 suggest that the translation process X_T may be inadequate to model X if the second order correlation function of this process needs to be represented accurately in addition to its marginal distribution and first order correlation functions.

6. Conclusions

A class of stationary non-Gaussian processes, referred to as the class of mixtures of translation processes, was defined by its finite dimensional distributions consisting of mixtures of finite dimensional distributions of translation processes. The class of mixtures of translation processes includes translation processes and is useful for both Monte Carlo simulation and analytical studies. As for translation processes, the mixture of translation processes can have a wide range of marginal distributions and correlation functions. Moreover, these processes can match a broader range of second order correlation functions than translation processes. The paper has also developed an algorithm for generating samples of any non-Gaussian process in the class of mixtures of translation processes. The algorithm is based on the sampling representation theorem for stochastic processes and properties of conditional distributions. Examples were presented to illustrate the proposed Monte Carlo algorithm and compare features of translation processes and mixture of translation processes.

References

[1] Cai GQ, Lin YK. Generation of non-Gaussian stationary stochastic processes. Phys Rev, E 1995;54(1):299–303.
[2] Cramer H, Leadbetter MR. Stationary and related stochastic processes. New York: Wiley; 1967.
[3] Grigoriu M. Simulation of stationary processes via a sampling theorem. J Sound Vib 1993;166(2):301–13.
[4] Grigoriu M. Applied non-Gaussian processes: examples, theory, simulation, linear random vibration, and MATLAB solutions. Englewoods Cliffs, NJ: Prentice-Hall; 1995.
[5] Grigoriu M. Non-Gaussian models in stochastic mechanics. Probab Engng Mech 2000;15(1):15–23.
[6] Grigoriu M. A class of non-Gaussian processes for Monte-Carlo simulation. J Sound Vib 2001;246(4):723–35.
[7] Grigoriu M, Balopoulou S. A simulation method for stationary Gaussian random functions based on the sampling theorem. Probab Engng Mech 1993;8(3/4):239–54.
[8] Grigoriu M, Lind NC. Optimal estimation of convolution integrals. J Engng Mech Div, ASCE 1980;106(EM6):1349–64.
[9] Gurley KR, Kareem A, Tognarelli MA. Simulation of a class of non-normal random processes. Int J Non-Linear Mech 1996;31(5):601–17.
[10] Johnson NL, Kotz S. Distributions in statistics: continuous multivariate distributions. New York: Wiley; 1972.