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Variability response functions for effective material properties

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ABSTRACT

The variability response function (*VRF*) is a well-established concept for efficient evaluation of the variance and sensitivity of the response of stochastic systems where properties are modeled by random fields that circumvents the need for computationally expensive Monte Carlo (MC) simulations. Homogenization of material properties is an important procedure in the analysis of structural mechanics problems in which the material properties fluctuate randomly, yet no method other than MC simulation exists for evaluating the variability of the effective material properties. The concept of a *VRF* for effective material properties is introduced in this paper based on the equivalence of elastic strain energy in the heterogeneous and equivalent homogeneous bodies. It is shown that such a *VRF* exists for the effective material properties of statically determinate structures. The *VRF* for effective material properties can be calculated exactly or by Fast MC simulation and depends on extending the classical displacement *VRF* to consider the covariance of the response displacement at two points in a statically determinate beam with randomly fluctuating material properties modeled using random fields. Two numerical examples are presented that demonstrate the character of the *VRF* for effective material properties, the method of calculation, and results that can be obtained from it.

1. Introduction

In many solid and structural mechanics problems the material properties exhibit random, non-periodic, spatial fluctuations. It is common in such problems to replace the randomly fluctuating material properties with a deterministic, homogenized set of material properties. The homogenized properties can be determined by direct averaging of the randomly fluctuating properties or by constraining the response of the homogenized system to be equivalent to that of the heterogeneous system in some sense such as having equivalent strain energy. Homogenization is particularly important in applying numerical techniques such as the finite element method to problems with randomly fluctuating material properties since such problems typically require very refined meshes to achieve high solution accuracy if the local material property fluctuations are to be represented. Homogenization is also a key part of upscaling material properties in the context of multi-scale analysis of solid mechanics problems (See, for one example, [1])

A key concept in homogenization is that of the representative volume element (RVE). The concept of the RVE is that, if material properties are homogenized over a sufficiently large volume, the resulting homogenized properties approach deterministic values. In other words, the variance of the estimate of the homogenized properties approaches zero as the homogenization volume increases. Practically speaking, of course, the volume of material in any solid mechanics problem is finite, and furthermore it may be desirable in some problems to retain medium to long range random fluctuations in the material properties by homogenizing over intermediate length scales. An example of the latter is evident in the approach of the moving window generalized method of cells which generates a smoothed version of random material property fluctuations to ease numerical analysis [2].

In homogenization problems in which the material volume is smaller than an RVE, it is critical to recognize that the homogeneous problem is itself stochastic, despite neglect of the spatial fluctuations of the material properties. This stochasticity arises from homogenizing over a finite volume so that the homogenized material properties are themselves random. An important issue, therefore, is the variance of the homogeneous, or effective, material properties which in turn defines the uncertainty in the response of the homogeneous system. Note that if homogenization occurs over an RVE this variance becomes zero and the problem is deterministic. If, on the other hand, the averaging volume is not an RVE, then the original stochastic problem that contains random spatially fluctuating material properties is replaced by a stochastic problem in which the material properties are spatially constant but random. Currently, the only practical way to evaluate the variance

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of the effective properties is through Monte Carlo (MC) simulation, which can be computationally very expensive. A method is introduced here for evaluating the variance of the effective properties that circumvents the need for brute force MC simulation by introducing a variability response function (*VRF*) for the effective properties that connects the spectral density of the randomly fluctuating material properties (that are modeled as homogeneous random fields) to the variance of the effective material properties.

A variability response function [3,4] describes the influence of the spectral content of a random material property field on the response of a structural system, usually the displacement. In cases where a *VRF* exists, the variance of the structural response can be calculated by a straightforward integration of the product of the spectral density of the material property field and the *VRF*, completely obviating the need for MC simulation or any other approximate technique such as perturbation or expansion methods. This results in very rapid evaluation of the response variance, and, perhaps more importantly, the possibility of evaluating the sensitivity of the response variability to the spectral content of the underlying material property field.

The body of this paper contains: (1) A general definition of the concept of the *VRF* for effective material properties and the proof that such a *VRF*, if it exists, depends on the *VRF* for the displacement variance and covariance; (2) proof of the existence of a *VRF* for the effective material properties of a statically determinate structure with a single applied point load and a corresponding numerical example; (3) proof of the existence of a *VRF* for the effective material properties of a statically determinate structure with an applied uniform distributed load and a corresponding numerical example. Item (3) requires the proof of the existence of a *VRF* for the covariance of the response displacement field of a statically determinate structure, which is also included.

2. Variability of effective properties

Consider a domain $\Omega \subset \mathbb{R}^3$ with volume V_Ω occupied by an isotropic elastic continuum characterized by a second order constitutive matrix $\mathbf{C}(\mathbf{x})$ and its inverse $\mathbf{D}(\mathbf{x}) = \mathbf{C}^{-1}(\mathbf{x})$. These matrices define the standard Hooke's law for three dimensional isotropic elastic solids

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x}) \tag{1}$$

where $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]^T$ and $\boldsymbol{\epsilon} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}]^T$ are the stress and strain vectors commonly used in engineering notation. In this section we consider only $\mathbf{C}(\mathbf{x})$ noting that $\mathbf{C}(\mathbf{x})$ and $\mathbf{D}(\mathbf{x})$ can be treated interchangeably since they completely define each other.

If the material occupying Ω is heterogeneous, $\mathbf{C}(\mathbf{x})$ is a function of position $\mathbf{x}$, and if the heterogeneity is random, then $\mathbf{C}(\mathbf{x})$ can be considered a matrix random field with mean matrix $\boldsymbol{\mu}$ that is independent of position and matrix of spectral density functions $\mathbf{S}_\mathbf{C}(\kappa_1, \kappa_2, \kappa_3)$ that depend on three wave numbers κ_1, κ_2, and κ_3 if $\mathbf{C}(\mathbf{x})$ is stationary.

Analysis of continuum mechanics problems in which the material properties are heterogeneous poses substantial challenges, and it is therefore often desirable to replace the heterogeneous material with a homogeneous material that is, in some sense, equivalent. This homogenization corresponds to replacing the matrix random field $\mathbf{C}(\mathbf{x})$ with the effective constitutive matrix

$$\bar{\mathbf{C}} = f(\mathbf{C}(\mathbf{x}), \Omega, \text{boundary conditions}) \tag{2}$$

which is a function not only of $\mathbf{C}(\mathbf{x})$ but also of the problem domain Ω and boundary conditions. The following discussion is restricted to fluctuations of the elastic modulus $E(\mathbf{x}) = C_{11}(\mathbf{x}) = C_{22}(\mathbf{x}) = C_{33}(\mathbf{x})$, noting that it could equivalently be applied to the shear modulus $G(\mathbf{x}) = C_{44}(\mathbf{x}) = C_{55}(\mathbf{x}) = C_{66}(\mathbf{x})$. It should be pointed out that the elastic modulus cannot fluctuate independently of the shear modulus and Poisson's ratio if the material is to remain locally isotropic.

The simplest definitions of the effective elastic modulus $\bar{E}$ are based on harmonic or arithmetic averaging of $E(\mathbf{x})$. These definitions give the Reuss and Voigt bounds on the effective modulus defined by

$$\bar{E}_r \le \bar{E} \le \bar{E}_v$$
$$\bar{E}_r = \langle E(\mathbf{x})^{-1}\rangle_\Omega = \frac{1}{V_\Omega}\left[\int_\Omega E(\mathbf{x})^{-1}\mathrm{d}V\right]^{-1}$$
$$\bar{E}_v = \langle E(\mathbf{x})\rangle_\Omega = \frac{1}{V_\Omega}\int_\Omega E(\mathbf{x})\mathrm{d}V. \tag{3}$$

The Voigt and Reuss bounds on the effective elastic modulus do not depend on the boundary conditions of the problem, and therefore the variances $\mathrm{var}[\bar{E}_v]$ and $\mathrm{var}[\bar{E}_r]$ depend directly on integrals of $E(\mathbf{x})$ that can be calculated using stochastic calculus [5] or estimated using MC simulation.

An alternative definition for the effective elastic modulus ensures that the elastic strain energy in the homogenized version of the problem is equivalent to that in the heterogeneous version. To use this definition the problem must be defined such that the solution of the homogeneous problem depends only on a single material constant, in this case assumed to be $\bar{E}$. Common examples include a uniaxial state of stress, pure shear, or a beam bending problem. For a set of traction boundary conditions $\mathbf{t}(\mathbf{x})$ that satisfy the condition that the response of the homogeneous problem depends only on a single elastic constant, and that result in a displacement field $\mathbf{u}(\mathbf{x})$ in the heterogeneous body, the equivalence of strain energy can be expressed as

strain energy of homogeneous problem
= strain energy of heterogeneous problem

$$g(\bar{E}, \Omega, \text{boundary conditions}) = \int_\Omega \boldsymbol{\epsilon}(\mathbf{x})\mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x})\mathrm{d}V = \int_{\partial\Omega} \mathbf{t}(\mathbf{x})\mathbf{u}(\mathbf{x})\mathrm{d}s. \tag{4}$$

The boundary conditions (bcs) and Ω are usually chosen so that an exact expression for $g(\bar{E}, \Omega, \text{bcs})$ is available in closed form, in which case Eq. (4) can be solved directly for $\bar{E}$. For example, if the boundary conditions are chosen to generate a uniaxial state of stress $\sigma_{11} = \sigma_0$, $g(\bar{E}, \Omega, \text{bcs}) = \int_\Omega \sigma_0^2/\bar{E}$ and if the problem is a cantilever beam with a point load applied at the free end, $g(\bar{E}, \Omega, \text{bcs}) = P^2L^3/3\bar{E}I$ with P being the applied load, L the length of the beam, and I the moment of inertia.

Assuming that $g(\bar{E}, \Omega, \text{bcs})$ is known and Eq. (4) can be solved for $\bar{E}$, the effective elastic modulus can be expressed as

$$\bar{E} = g^*\left(\Omega, \text{bcs}, \int_{\partial\Omega}\mathbf{t}(\mathbf{x})\mathbf{u}(\mathbf{x})\mathrm{d}s\right). \tag{5}$$

In many cases, the effect of Ω and of the boundary conditions will appear as a deterministic coefficient C_1, so that the variance of the effective elastic modulus can be expressed as

$$\mathrm{var}[\bar{E}] = C_1^2\,\mathrm{var}\left[\int_{\partial\Omega}\mathbf{t}(\mathbf{x})\mathbf{u}(\mathbf{x})\mathrm{d}s\right]. \tag{6}$$

This expression suggests that the variance of the effective elastic modulus depends on the variance of an integral of the displacement field. Furthermore, it is known that in certain cases the variance of the displacement depends on a variability response function through the following integral expression

$$\mathrm{var}[u_i(\mathbf{x})] = \iiint_{-\infty}^{\infty} S_f(\kappa_1, \kappa_2, \kappa_3)VRF_{u_i}(\mathbf{x}, \kappa_1, \kappa_2, \kappa_3)\mathrm{d}\kappa_1\mathrm{d}\kappa_2\mathrm{d}\kappa_3 \tag{7}$$

where $S_f(\kappa_1, \kappa_2, \kappa_3)$ is the spectral density function describing the spatially random fluctuations of the material property field and $VRF_{u_i}(\mathbf{x}, \kappa_1, \kappa_2, \kappa_3)$ is the deterministic variability response function. Note that this relationship has been proven to exist only for statically determinate problems in one dimension [3,4], although it is written here in the more general three dimensional form. Note also that it has been shown to be approximately valid for certain statically indeterminate problems in one and two dimensions [6–9]. The existence of a *VRF* for the displacement suggests that a *VRF* may exist for the effective elastic modulus such that

$$\mathrm{var}[\bar{E}] = \iiint_{-\infty}^{\infty} S_f(\kappa_1, \kappa_2, \kappa_3) VRF_{\bar{E}}(\kappa_1, \kappa_2, \kappa_3)\mathrm{d}\kappa_1\mathrm{d}\kappa_2\mathrm{d}\kappa_3 \tag{8}$$

and that it may be possible to express $VRF_{\bar{E}}(\kappa_1, \kappa_2, \kappa_3)$ in terms of $VRF_u(\mathbf{x}, \kappa_1, \kappa_2, \kappa_3)$. Verifying these last two statements, that a *VRF* exists for the effective material properties, and that this *VRF* depends on the *VRF* of the response displacements, are the two main objectives of this paper, and they are now investigated for some specific structural mechanics problems.

3. Effective flexibility of beams

For structural beams it is convenient to consider uncertainty in the flexibility in the form

$$D(x) = \frac{1}{EI(x)} = \frac{1}{EI_0}(1 + f(x)) \tag{9}$$

where EI_0 is the nominal stiffness and $f(x)$ is a zero mean homogeneous random field with spectral density function $S_f(\kappa)$ that is a function of the wave number κ. The variance of the response transverse displacement $u(x)$ of such beams can be expressed in the following integral form involving a *VRF*

$$\mathrm{var}[u(x)] = \int_{-\infty}^{\infty} VRF_u(x, \kappa)S_f(\kappa)\mathrm{d}\kappa. \tag{10}$$

Note that this expression is exact for statically determinate beams but only approximate for statically indeterminate beams. The equivalent homogeneous beam has the same geometry and boundary conditions, but has $D(x)$ replaced with the effective flexibility $\bar{D}$ (constant along the length of the beam) which plays the role of $\bar{E}$ in the previous discussions.

4. Statically determinate beams with a single point load

Consider a statically determinate beam of length L in which the flexibility varies in the way described in Eq. (9) and which is loaded with a single point load P applied at position x_P. Eq. (4) expressed for such a beam is

$$g(\bar{D}, L, \mathrm{bcs}) = C_1P^2\bar{D} = Pu(x_P) = \int_{\partial\Omega} \mathbf{t}(\mathbf{x})\mathbf{u}(\mathbf{x})\mathrm{d}s \tag{11}$$

which can be solved for

$$\bar{D} = \frac{1}{C_1P}u(x_P) \tag{12}$$

where $u(x_P)$ is the transverse displacement of the randomly heterogeneous beam at location x_P and C_1 is a deterministic constant depending on the length L and the boundary conditions. The variance of the effective flexibility can then be expressed as

$$\begin{aligned}\mathrm{var}[\bar{D}] &= \left(\frac{1}{C_1P}\right)^2 \mathrm{var}[u(x_P)] \\ &= \int_{-\infty}^{\infty} S_f(\kappa)\left(\frac{1}{C_1P}\right)^2 VRF_u(x_P, \kappa)\mathrm{d}\kappa \\ &= \int_{-\infty}^{\infty} S_f(\kappa)VRF_{\bar{D}}(\kappa)\mathrm{d}\kappa \end{aligned} \tag{13}$$

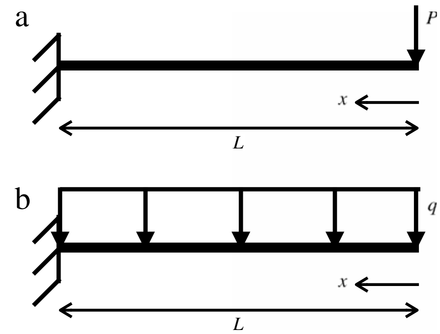

Fig. 1. Cantilever beams with point and distributed loads.

with

$$VRF_{\bar{D}}(\kappa) = \left(\frac{1}{C_1P}\right)^2 VRF_u(x_P, \kappa). \tag{14}$$

Consequently, for a statically determinate beam with a single applied point load the *VRF* of the effective flexibility has the same functional form as the *VRF* of the displacement, but is multiplied by a coefficient that depends on the applied point load, the geometry of the beam, and the boundary conditions.

4.1. Example

The beam shown in Fig. 1(a) is a statically determinate cantilever with a single point load P applied at the free end $x_P = 0$. For this particular beam

$$g(\bar{D}, L, \mathrm{bcs}) = \frac{P^2L^3\bar{D}}{3} \tag{15}$$

as $C_1 = L^3/3$. Using Eq. (12), the expression for $\bar{D}$ is

$$\bar{D} = \frac{3u(0)}{PL^3}. \tag{16}$$

It should be noted that as $u(0)$ is the tip deflection of the randomly heterogeneous beam, $u(0)$ is a random variable and consequently $\bar{D}$ is also a random variable.

Eq. (14) leads to the following expression relating the *VRF* of the displacement to the *VRF* of the effective flexibility

$$VRF_{\bar{D}}(\kappa) = \left(\frac{3}{PL^3}\right)^2 VRF_u(0, \kappa). \tag{17}$$

Let $P = 16{,}000$ N, $L = 16$ m and $EI_0 = 1.25 \times 10^7$ N m^2. The variability response function $VRF_u(0, \kappa)$ resulting from these numerical parameters is shown in Fig. 2. It has been estimated by Fast Monte Carlo (FMC) simulation [10,11] though it can be calculated from a closed form expression following the procedures in [3,4,10]. The corresponding variability response function $VRF_{\bar{D}}(\kappa)$ is shown in Fig. 3, has the same shape as $VRF_u(\kappa)$ and differs only by the constant $\left(3/PL^3\right)^2 = 2.09 \times 10^{-15}$.

For demonstration purposes, consider the two spectral density functions

$$\begin{aligned} S_{f,1}(\kappa) &= \frac{2\sigma_f^2\beta}{\pi(\kappa^2 + \beta^2)} \\ S_{f,2}(\kappa) &= \begin{cases} 0.033 & 0 \le \kappa \le 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{18}$$

with $\sigma_f^2 = 0.1$ and $\beta = 0.25$ that define the homogeneous random field $f(x)$. Both spectral densities correspond to the same variance $\sigma_f^2 = 0.1$. Fig. 4 displays a few samples of $D(x)$ resulting from

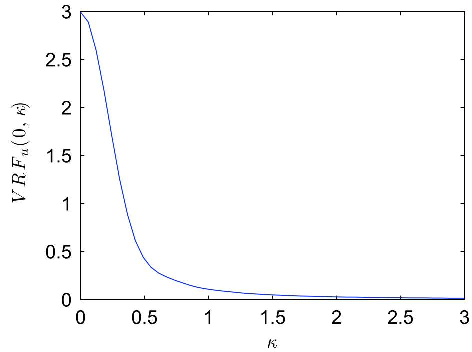

Fig. 2. VRF for cantilever tip displacement (example in Section 4.1).

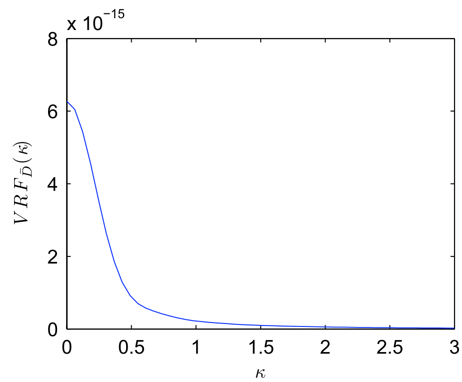

Fig. 3. VRF for effective flexibility (example in Section 4.1).

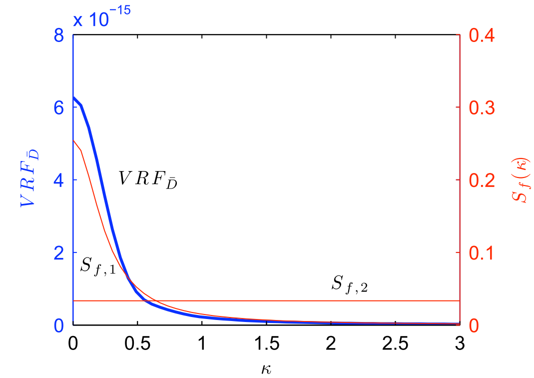

Fig. 5. VRF for effective flexibility shown with spectral densities of flexibility field (example in Section 4.1).

each of the spectra under a truncated Gaussian assumption for the marginal PDF. It should be noted that the results are invariant to the choice of a marginal PDF, however, the marginal PDF should respect physical restrictions in the numerical values allowable for the flexibility (consequently the Gaussian is not acceptable as a marginal PDF). Fig. 5 shows the two spectra on the same plot with $VRF_{\bar{D}}(\kappa)$ in order to visualize the corresponding overlaps. Eq. (13) provides the following estimates for the variance of $\bar{D}$ for the two spectral densities: $\text{var}[\bar{D}] = 3.2 \times 10^{-16}$ for $S_{f,1}(\kappa)$ and $\text{var}[\bar{D}] = 7.2 \times 10^{-17}$ for $S_{f,2}(\kappa)$. These values agree well with results of direct MC simulations of the beam response with 500 samples which yield $\text{var}[\bar{D}] = 3.1 \times 10^{-16}$ and $\text{var}[\bar{D}] = 7.0 \times 10^{-17}$, respectively. Note that the *VRF*s use here for the effective flexibility are exact, and any discrepancy between the prediction of the *VRF* approach (Eq. (13)) and the results of MC simulations stems from three potential sources of numerical error: (1) estimation error associated with the finite number of samples used in the MC simulations; (2) estimation error associated with the finite number of samples used in the estimation of the *VRF*; (3) error associated with the numerical integration of the product of the *VRF* and the spectral density to predict the effective property variance (Eq. (13)). Each of these errors could be reduced to achieve an arbitrary degree of accuracy provided sufficient time and computational resources were available.

The effective flexibility as defined in this work differs from the arithmetic and harmonic averages of $D(x)$. For the same

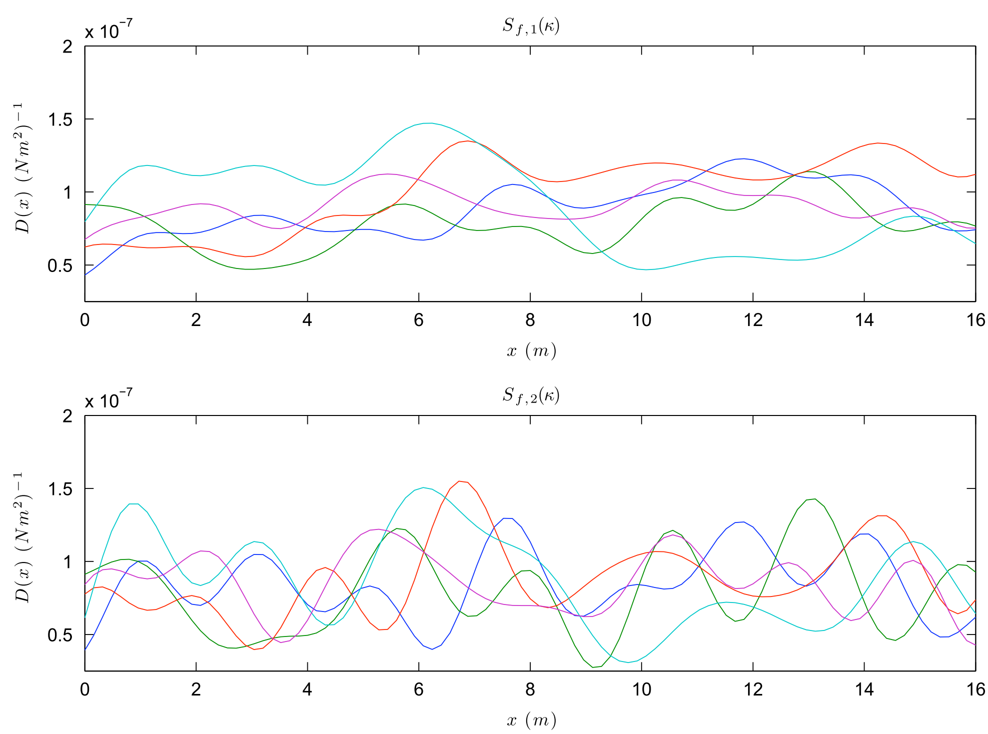

Fig. 4. Samples of $\mathbf{D}(\mathbf{x})$ generated using spectral densities $S_{f,1}(\kappa)$ and $S_{f,2}(\kappa)$, both with the same marginal PDF (truncated Gaussian) and variance $\sigma_f^2 = 0.1$.

simulations described above, the variance of the arithmetic mean of $D(x)$ (Reuss) and of the harmonic mean (Voigt) are 2.4×10^{-16} and 3.3×10^{-16} for $S_{f,1}(\kappa)$ and 4.0×10^{-17} and 2.9×10^{-15} for $S_{f,2}(\kappa)$. Note that since flexibility is considered in this case, the Voigt average corresponds to a harmonic average of $D(x)$ whereas the Reuss average corresponds to an arithmetic average of $D(x)$. In certain cases the variance of the Voigt and Reuss averages differs substantially from the variance of the strain energy based effective flexibility.

5. Statically determinate beams with uniform load

Consider now a statically determinate beam identical to that introduced in the previous section but with a deterministic uniform distributed load q applied along the entire length of the beam. For this case

$$g(\bar{D}, L, \text{bcs}) = \int_0^L q\bar{u}(x)\mathrm{d}x = q^2 C_2 \bar{D} \tag{19}$$

where $\bar{u}(x)$ denotes the transverse displacement of the homogeneous beam and the constant C_2 represents the influence of the beam's geometry and boundary conditions, as before. The right-hand side of Eq. (4) is written as

$$\int_{\partial\Omega} t(x)u(x)\mathrm{d}s = \int_0^L qu(x)\mathrm{d}x \tag{20}$$

where $u(x)$ denotes the transverse displacement of the randomly heterogenous beam. Combining now Eqs. (4), (19) and (20) leads to the following expression for $\bar{D}$

$$\bar{D} = \frac{1}{qC_2}\int_0^L u(x)\mathrm{d}x. \tag{21}$$

The variance of $\bar{D}$ can be written as

$$\mathrm{var}[\bar{D}] = \left(\frac{1}{qC_2}\right)^2 \mathrm{var}\left[\int_0^L u(x)\mathrm{d}x\right] = \left(\frac{1}{qC_2}\right)^2 \iint_0^L c(u(x_1), u(x_2))\mathrm{d}x_1\mathrm{d}x_2 \tag{22}$$

where $c(u(x_1), u(x_2))$ is the covariance function of $u(x)$ (The second part of Eq. (22) is derived from the first part by direct calculations following equations K.1 and K.2 in [12]). The expression in Eq. (22) suggests that if a *VRF* exists such that

$$c(u(x_1), u(x_2)) = \int_{-\infty}^{\infty} S_f(\kappa) VRF_{u_1u_2}(x_1, x_2, \kappa)\mathrm{d}\kappa \tag{23}$$

with $S_f(\kappa)$ being the spectral density of random field $f(x)$, then

$$\begin{aligned}\mathrm{var}[\bar{D}] &= \left(\frac{1}{qC_2}\right)^2 \iint_0^L \int_{-\infty}^{\infty} S_f(\kappa) VRF_{u_1u_2}(x_1, x_2, \kappa)\mathrm{d}\kappa\,\mathrm{d}x_1\mathrm{d}x_2 \\ &= \int_{-\infty}^{\infty} S_f(\kappa)\left[\left(\frac{1}{qC_2}\right)^2 \iint_0^L VRF_{u_1u_2}(x_1, x_2, \kappa)\mathrm{d}x_1\mathrm{d}x_2\right]\mathrm{d}\kappa \\ &= \int_{-\infty}^{\infty} S_f(\kappa) VRF_{\bar{D}}(\kappa)\mathrm{d}\kappa \end{aligned} \tag{24}$$

with

$$VRF_{\bar{D}}(\kappa) = \left[\left(\frac{1}{qC_2}\right)^2 \iint_0^L VRF_{u_1u_2}(x_1, x_2, \kappa)\mathrm{d}x_1\mathrm{d}x_2\right] \tag{25}$$

What differs significantly from the previous case involving a point load is that $VRF_{\bar{D}}(\kappa)$ is now a function of integrals of *VRF*s.

The existence of $VRF_{u_1u_2}(x_1, x_2, \kappa)$ will now be established. This in turn proves the existence of $VRF_{\bar{D}}(\kappa)$ for statically determinate beams loaded with a uniform distributed load. The displacement field can be expressed as

$$u(x) = -D_0\int_0^x h(x, \xi)M(\xi)[1 + f(x)]\mathrm{d}\xi \tag{26}$$

where D_0 is the nominal value of the flexibility (Eq. (9)), $h(x, \xi)$ is the deterministic Green's function and $M(\xi)$ is the bending moment, which is independent of the flexibility and deterministic since the beam is statically determinate. The correlation function $r(u(x), u(y))$ is easily computed using Eq. (26),

$$\begin{aligned} r(u(x_1), u(x_2)) &= E[u(x_1)u(x_2)] \\ &= D_0^2\int_0^{x_1}\int_0^{x_2} h(x_1, \xi_1)h(x_2, \xi_2)M(\xi_1)M(\xi_2) \\ &\quad\times\left[1 + R_f(\xi_1 - \xi_2)\right]\mathrm{d}\xi_1\mathrm{d}\xi_2 \end{aligned} \tag{27}$$

where $R_f(\xi_1 - \xi_2)$ is the autocorrelation of the homogeneous, zero mean random field $f(x)$. The covariance function $c(u(x_1)u(x_2))$ is therefore written as

$$\begin{aligned} c(u(x_1), u(x_2)) = D_0^2\int_0^{x_1}\int_0^{x_2} h(x_1, \xi_1)h(x_2, \xi_2)M(\xi_1) \\ \times M(\xi_2)R_f(\xi_1 - \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2. \end{aligned} \tag{28}$$

If the spectral density $S_f(\kappa)$ is substituted for $R_f(\xi_1 - \xi_2)$ through the Wiener–Khintchine transform, the covariance function of $u(x)$ becomes

$$\begin{aligned} c(u(x_1), u(x_2)) = D_0^2\int_0^{x_1}\int_0^{x_2} h(x_1, \xi_1)h(x_2, \xi_2)M(\xi_1) \\ \times M(\xi_2)\int_{-\infty}^{\infty} S_f(\kappa)\mathrm{e}^{\mathrm{i}\kappa(\xi_1-\xi_2)}\mathrm{d}\xi_1\mathrm{d}\xi_2. \end{aligned} \tag{29}$$

Changing the order of integration yields

$$c(u(x_1), u(x_2)) = \int_{-\infty}^{\infty} VRF_{u_1u_2}(x_1, x_2, \kappa)S_f(\kappa)\mathrm{d}\kappa. \tag{30}$$

where the *VRF* is defined as

$$\begin{aligned} VRF_{u_1u_2}(x_1, x_2, \kappa) = D_0^2\int_0^{x_1}\int_0^{x_2} h(x_1, \xi_1)h(x_2, \xi_2) \\ \times M(\xi_1)M(\xi_2)\mathrm{e}^{\mathrm{i}\kappa(\xi_1-\xi_2)}\mathrm{d}\xi_1\mathrm{d}\xi_2. \end{aligned} \tag{31}$$

Eqs. (30) and (31) are of interest for several reasons. First, they show that a *VRF* exists for the covariance function of the non-homogeneous, random field $u(x)$. Second, the derivation of this *VRF* is independent of the marginal distribution of $f(x)$. Third $VRF_{u_1u_2}(x_1, x_2, \kappa)$ is defined on the two-dimensional domain (x_1, x_2).

In principal $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ can be calculated exactly using closed form expressions for the Green's and moment functions $h(x, \xi)$ and $M(\xi)$, but in practice it is often more convenient to estimate the *VRF* using the very efficient Fast Monte Carlo (FMC) method [10,11]:

1. Select the variance σ_f^2 of the flexibility fluctuations.
2. Select the number of simulations N_{sim} used to estimate each value of the *VRF*.
3. Fix values of x_1 and x_2.
4. Fix the value of κ.
5. Generate N_{sim} realizations of $f(x)$ with $j = 1, 2, \ldots, N_{sim}$:

$$f(x) = \sqrt{2}\sigma_f\cos\left(\kappa x + \theta^{(j)}\right), \qquad \theta^{(j)} = \left(j - \frac{1}{2}\right)\left(\frac{2\pi}{N_{sim}}\right) \tag{32}$$

6. For each realization of $f(x)$ compute $u^{(j)}(x_1)$ and $u^{(j)}(x_2)$.
7. Estimate the covariances using

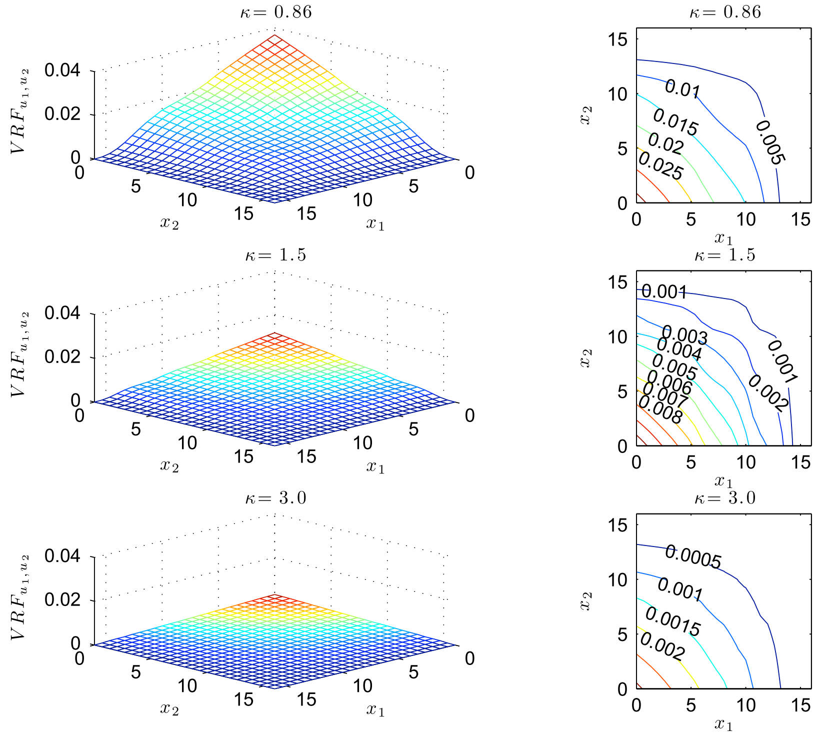

Fig. 6. $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ shown as surface and contour plots for fixed values of κ.

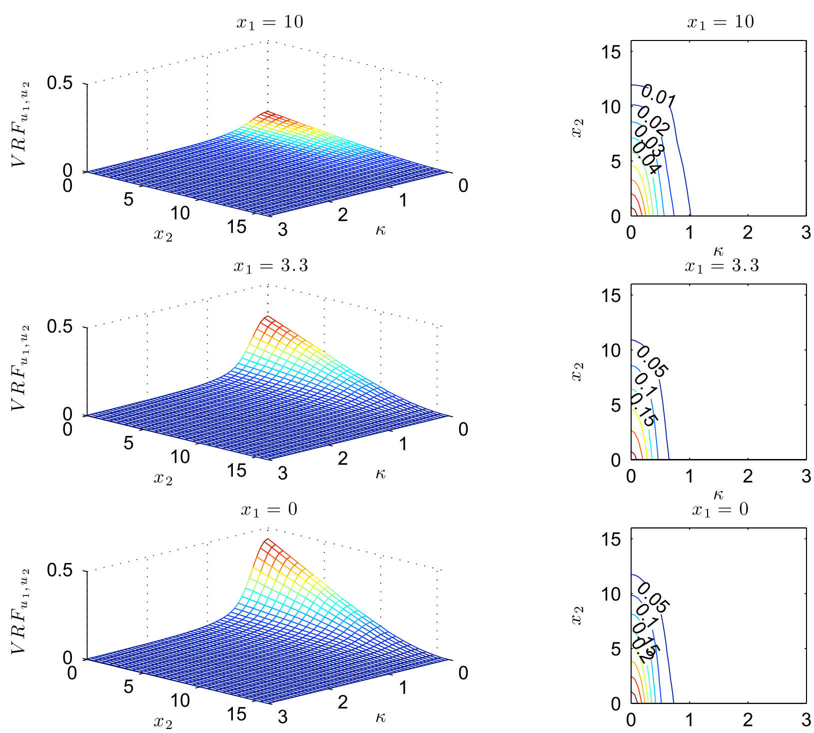

Fig. 7. $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ shown as surface and contour plots for fixed values of x_1 (example in Section 5.1).

$$c(u(x_1), u(x_2)) \approx \frac{1}{N_{sim}} \sum_{j=1}^{N_{sim}} \left(u^{(j)}(x_1) - \hat{u}(x_1)\right)\left(u^{(j)}(x_2) - \hat{u}(x_2)\right)$$

$$\hat{u}(x_1) \approx \frac{1}{N_{sim}} \sum_{j=1}^{N_{sim}} u^{(j)}(x_1)$$

$$\hat{u}(x_2) \approx \frac{1}{N_{sim}} \sum_{j=1}^{N_{sim}} u^{(j)}(x_2) \tag{33}$$

8. Calculate

$$VRF_{u_1,u_2}(x_1, x_2, \kappa) = \frac{c(u(x_1), u(x_2))}{\sigma_f^2} \tag{34}$$

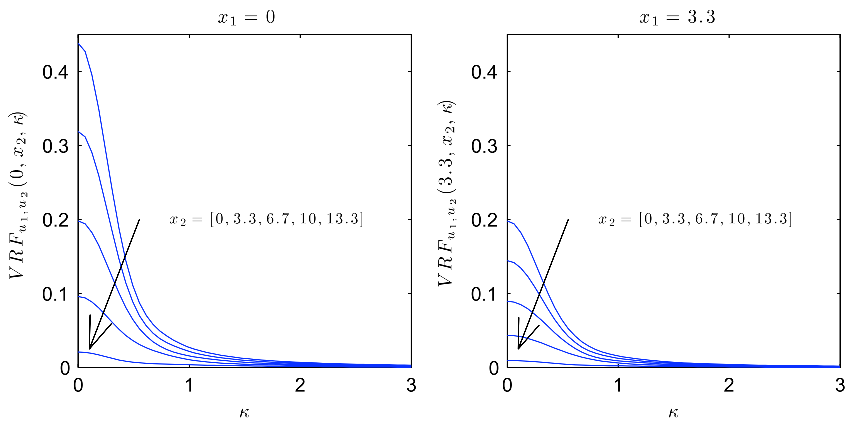

Fig. 8. $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ shown for fixed values of x_1 and x_2 (example in Section 5.1).

9. Repeat steps 4–8 for a number of equidistant values of κ on the wave number domain from zero to an appropriately selected upper cutoff wave number κ_u beyond which the values of the *VRF* become negligibly small.
10. Repeat steps 3–9 for a number of equidistant values of x_1 and x_2 on a two-dimensional grid in the $0 \leq x_1, x_2 \leq L$ domain.

Once $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ is estimated by this FMC procedure, $VRF_{\bar{D}}(\kappa)$ is readily obtained by numerical integration following Eq. (25), or, alternatively, $VRF_{\bar{D}}(\kappa)$ can be estimated directly by an FMC algorithm similar to that described above.

5.1. Example

Consider now the statically determinate cantilever beam in Fig. 1(b) with a uniform load q applied over the entire length of the beam. For the homogeneous beam, the displacement is given by

$$\bar{u}(x) = \frac{q\bar{D}}{24}\left(x^4 - 4L^3x + 3L^4\right) \tag{35}$$

which yields, according to Eq. (19), $C_2 = L^5/20$. Eq. (22) is then written as

$$\mathrm{var}[\bar{D}] = \left(\frac{20}{qL^5}\right)^2 \mathrm{var}\left[\int_0^L u(x)\mathrm{d}x\right] \tag{36}$$

and Eq. (25) as

$$VRF_{\bar{D}}(\kappa) = \left[\left(\frac{20}{qL^5}\right)^2 \int_0^L \int_0^L VRF_{u_1,u_2}(x_1, x_2, \kappa)\mathrm{d}x_1\mathrm{d}x_2\right]. \tag{37}$$

Let $q = 1000$ N/m, $L = 16$ m and $EI_0 = 1.25 \times 10^7$ N m^2, and consider again the two spectral density functions defined by Eq. (18). Using the FMC algorithm described above, $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ has been estimated using $N_{sim} = 50$. FMC simulations were carried out for 25×25 values of x_1 and x_2 evenly spaced in the two-dimensional interval [0, 16], and for $n_\kappa = 50$ values of κ evenly spaced in the interval [0, 3]. Consequently, the complete procedure requires the generation of $n_\kappa N_{sim} = 2500$ samples of $D(x)$ according to Eqs. (9) and (32) and the subsequent solution for $u(x)$ for each one of these samples. Using the finite element method to calculate $u(x)$ with 100 Euler–Bernoulli beam elements, the entire FMC estimation requires approximately 75 seconds when executed in MATLAB on a MacBook Pro laptop computer with a 2.6 GHz Intel Core 2 Duo processor and 4 GB of RAM.

The FMC algorithm provides an estimate of $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ which depends on three input arguments, making it difficult to visualize. Fig. 6 shows $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ as surface and contour plots for $\kappa = \{0.86, 1.5, 3.0\}$. This figure shows that the general shape of the *VRF* is similar at different values of κ, with the peak value always occurring at $x_1 = x_2 = 0$, corresponding to the tip of the beam. As κ increases towards its maximum value of 3.0, the magnitude of the VRF decreases, which is in agreement with the general form of $VRF_u(\kappa)$ (Fig. 2) which has a maximum at $\kappa = 0$ and approaches zero as κ increases.

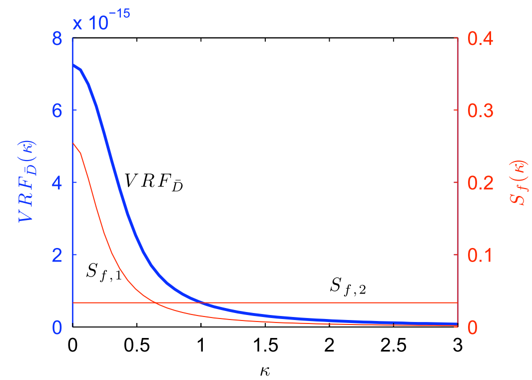

Fig. 9. $VRF_{\bar{D}}(\kappa)$ shown with spectral densities $S_{f,1}(\kappa)$ and $S_{f,2}(\kappa)$ (example in Section 5.1).

In Fig. 7, sections through $VRF_{u_1,u_2}(x_1, x_2, \kappa)$ at fixed values of x_1 are shown. Sections through these surfaces at constant x_2 are shown in Fig. 8, and show a shape similar to that in Fig. 2, with the magnitude of the *VRF* decreasing as $|x_1 - x_2|$ increases.

Fig. 9 shows $VRF_{\bar{D}}(\kappa)$, calculated using Eq. (37), together with the two spectral density functions of Eq. (18). Numerical integration of the third line of Eq. (24) with $S_{f,1}(\kappa)$ gives $\mathrm{var}[\bar{D}] = 4.3 \times 10^{-16}$ and direct MC simulations of the beam's response with 1000 samples gives 4.4×10^{-16}, a reasonably good agreement. For $S_{f,2}(\kappa)$ the agreement is also good, with the *VRF* approach giving 1.2×10^{-16} and direct MC simulations yielding 1.2×10^{-16}. Note again that the *VRF* s used here for the effective flexibility are exact, and any difference between the *VRF*-based predictions for the variance of $\bar{D}$ and the corresponding MC results is merely an artifact of various estimation and numerical integration errors that can be reduced to arbitrarily small values given sufficient computational resources.

6. Conclusion

A variability response function (*VRF*) has been introduced for the effective elastic material properties of a heterogeneous body. This *VRF* depends directly on the boundary conditions and geometry of the body, as well as on the *VRF* for the displacement response, when the homogenization of the medium is defined based on equivalence of elastic strain energy in the heterogeneous and homogeneous bodies. For the case of a statically determinate beam subject to either a single point load or a uniformly distributed load the *VRF* for the effective flexibility exists and can be calculated by scaling the displacement *VRF* (point load case) or scaling and integrating the *VRF* for the covariance of displacements along the length of the beam (uniform load case). In order to demonstrate the second result, the existence of a *VRF* for the covariance of displacements at points along a statically determinate beam has been proven. Two numerical examples demonstrate the efficacy of the *VRF* approach for predicting the variance of the effective flexibility. The *VRF* for effective material properties introduced here is useful for efficiently evaluating the variance of effective material properties averaged over finite material volumes and also for conducting studies on the sensitivity of the effective property variability to the spectral contents of the underlying material property random field. The calculation of the *VRF* is done using a very efficient Fast Monte Carlo algorithm. Once the *VRF* is calculated, the evaluation of effective material property variability and sensitivity can be accomplished without recourse to any additional MC simulation, requiring only simple numerical integration of a one dimensional integral in the wave number domain.

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