A High Throughput Polynomial and Rational Function Approximations Evaluator

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Joint work with **N. Brisebarre, G. Constantinides, M. Ercegovac, M. Iștoan & J-M. Muller**

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Context

Mathematical function evaluation

How?

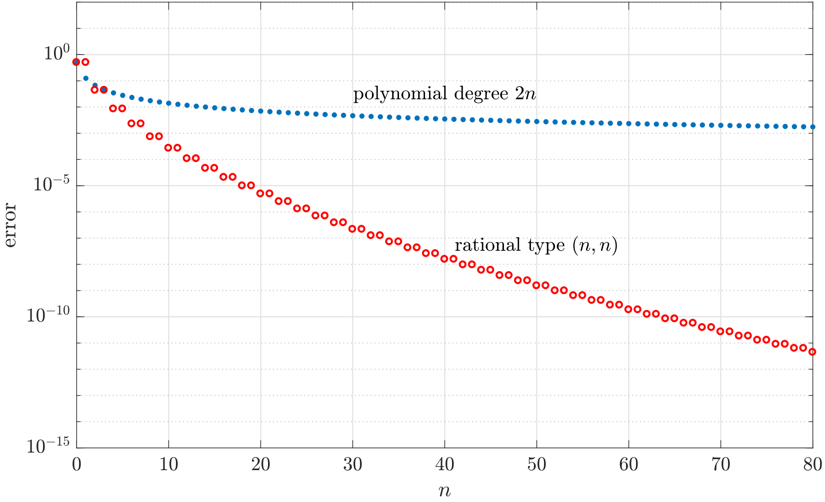
- in particular cases, use *ad hoc* solutions
  - e.g. CORDIC, tabulate-and-compute algorithms
- in general, polynomial and rational function approximations

Rational functions are more powerful than polynomials

Polynomials vs rational approximations

Celebrated example in approximation theory: $f(x) = |x|, x \in [-1, 1]$

- polynomial $2n$: $O(1/n)$ [Bernstein 1910s, Varga & Carpenter 1985]
- rational (n, n): $O(\exp(-\pi\sqrt{n}))$ [Newmann 1964, Stahl 1993]



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Our goal:

Benefits to implementing rational functions in HW?

The E-method [Ercegovac 1975, 1977]

$$R(x) = \frac{p_\mu x^\mu + p_{\mu-1}x^{\mu-1} + \cdots + p_0}{q_\nu x^\nu + q_{\nu-1}x^{\nu-1} + \cdots + 1}$$

Idea:

- $R(x)$ mapped to a linear system: $\mathbf{A}_x \mathbf{y} = \mathbf{p}$

$$\begin{bmatrix} 1 & -x & 0 & & \cdots & 0 \\ q_1 & 1 & -x & 0 & \cdots & 0 \\ q_2 & 0 & 1 & -x & \cdots & 0 \\ & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 0 \\ q_{n-1} & & & & 1 & -x \\ q_n & & & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}$$

$$y_0 = R(x)$$

$$n = \max\{\mu, \nu\}, p_k = 0, \mu < k \leqslant n, q_k = 0, \nu < k \leqslant n$$

The E-method [Ercegovac 1975, 1977]

- digit-by-digit computation

$$\mathbf{w}^{(j)} = r \cdot \left[\mathbf{w}^{(j-1)} - \mathbf{A}_x \mathbf{d}^{(j-1)}\right], \qquad j = 1, \ldots, m$$

m - number of iterations
$\mathbf{d}^{(0)} = \mathbf{0}, \mathbf{w}^{(0)} = \mathbf{p}$
$d_i^{(j)}$ - single digit rounding of $w_i^{(j)}$

Result: m-digit solution in radix r

$$y_k = \sum_{j=1}^{m} d_k^{(j)} r^{-j}$$

Rule of thumb: *comparable* cost for deg n poly & type (n, n) rational

Convergence of the E-method

$\to$ requires *bounds* on the p_k's, q_k's and approximation domain $[a, b]$

$$\begin{cases} \forall k, |p_k| \leq \xi, \\ \forall k, |x| + |q_k| \leq \alpha \end{cases}$$

$$\xi = \frac{1}{2}(1 + \mathit{\Delta}),$$

$$0 < \mathit{\Delta} < 1,$$

$$\alpha \leq (1 - \mathit{\Delta})/(2r)$$

- **polynomial**: always (up to scaling the p_k's and change of variable)
- **rational**: *requires* bounded $|q_k|$'s $\to$ E-fraction approximations

Computing E-fraction approximations

Input: $f \in \mathcal{C}([a,b]), \mu, \nu \in \mathbb{N}$, magnitude bound $d > 0$.

Output: $R(x) = \dfrac{p_\mu x^\mu + p_{\mu-1}x^{\mu-1} + \cdots + p_0}{q_\nu x^\nu + q_{\nu-1}x^{\nu-1} + \cdots + 1}$, with $\max_{1 \leqslant k \leqslant \nu} |q_k| \leqslant d$,
s.t.

$$\max_{x \in [a,b]} |f(x) - R(x)|$$

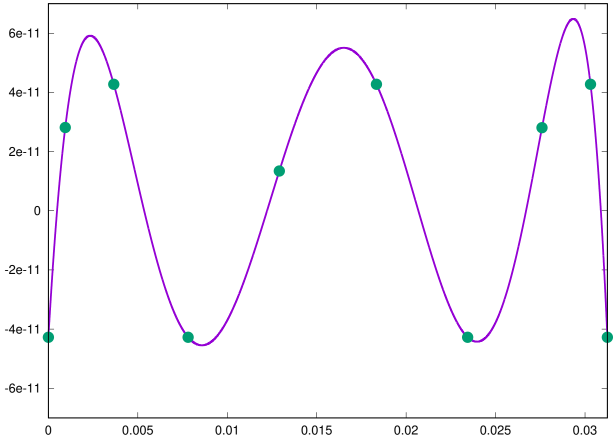
is **minimal**.

How?

- $\nu = 0 \rightarrow$ use the polynomial Remez exchange algorithm [Remez 1934]
- $\nu > 0 \rightarrow$ we ***developed*** a greedy-based iterative algorithm

An example

Ex. 1: $f(x) = \sqrt{1 + (9x/2)^4}, x \in [0, 1/32]$, $(\mu, \nu) = (4, 4)$
$d = 3/16$ ($\Delta = 1/8$ and $r = 2$)
Iteration 1



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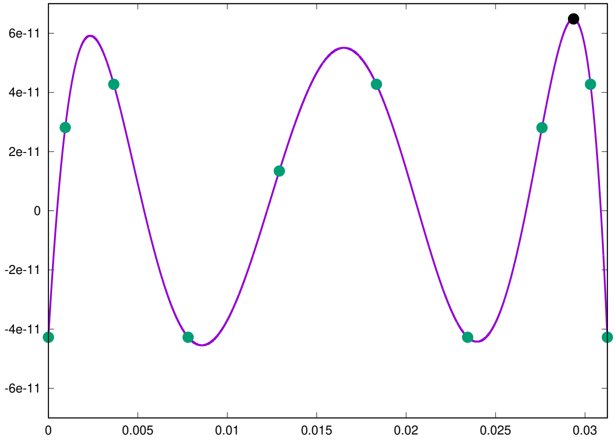

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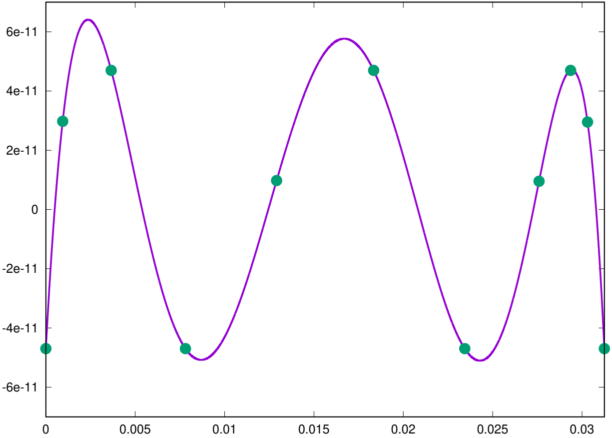
Iteration 2



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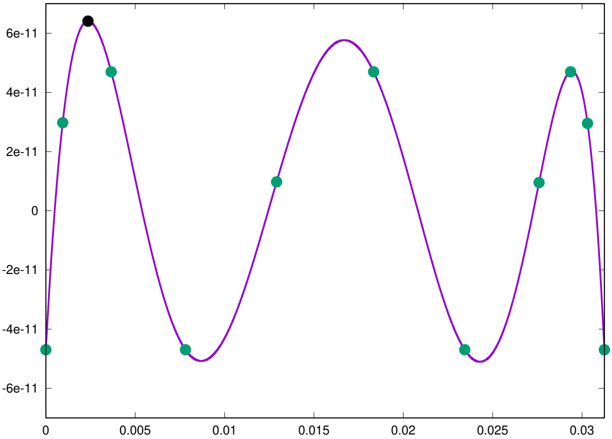

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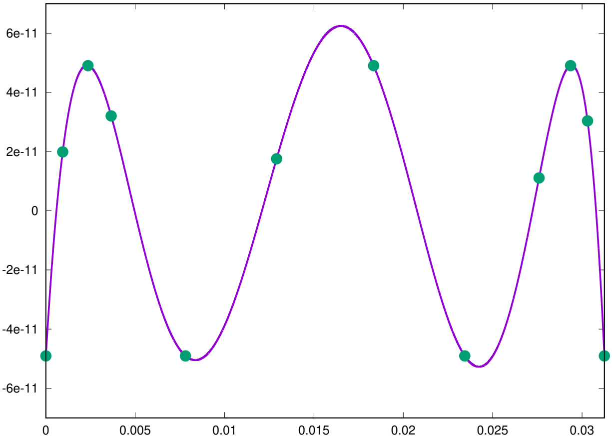
Iteration 3



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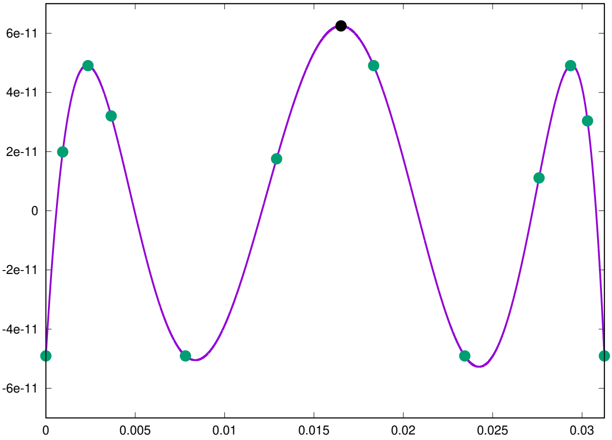

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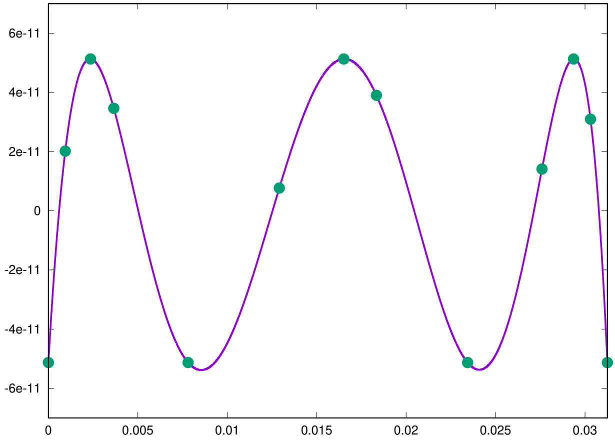
Iteration 4



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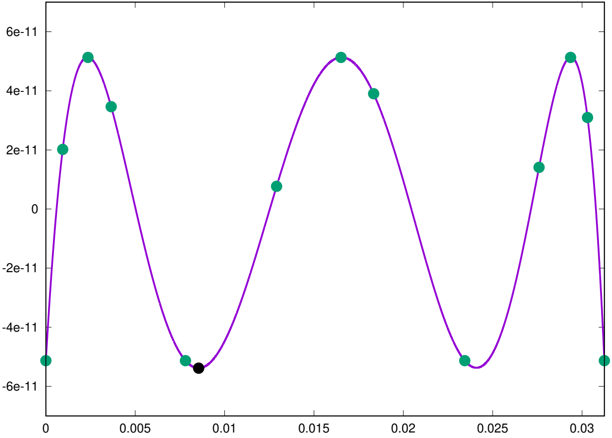

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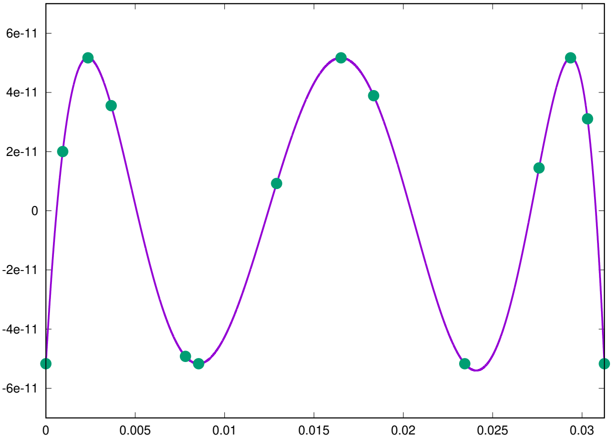

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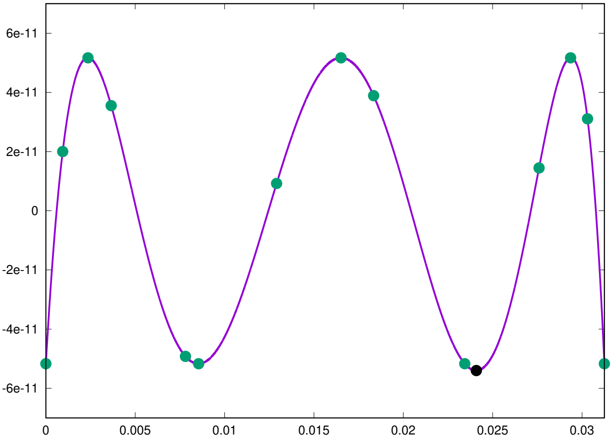

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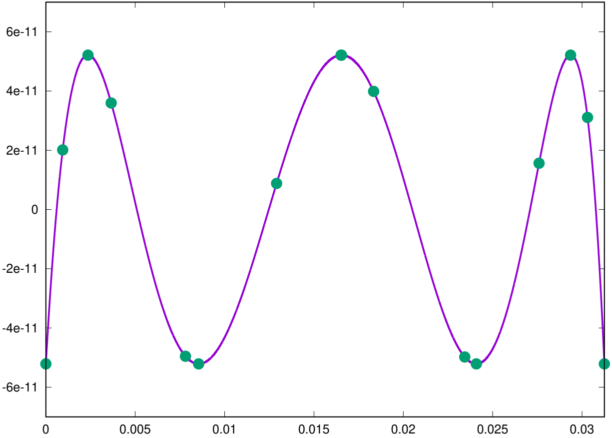
Iteration 6



Approximation error $\simeq \mathbf{5.21 \cdot 10^{-11}}$
Polynomial $(\mu, \nu) = (4, 0)$ error $\simeq \mathbf{3.41 \cdot 10^{-10}}$

Machine coefficient E-fractions

In practice, finite precision

$\rightarrow$ we target fixed-point implementations

- target error $\leqslant 2^{-m}, m \in \mathbb{N}$
- coefficients of the form $i/2^m, -2^m \leqslant i \leqslant 2^m$

Ex. 1: $m = 32$, target error $2^{-m} \simeq 2.33 \cdot 10^{-10}$

- real-coefficient error $5.21 \cdot 10^{-11}$
- rounding error $\mathbf{1.11 \cdot 10^{-9}} > 2^{-m}$ ☹

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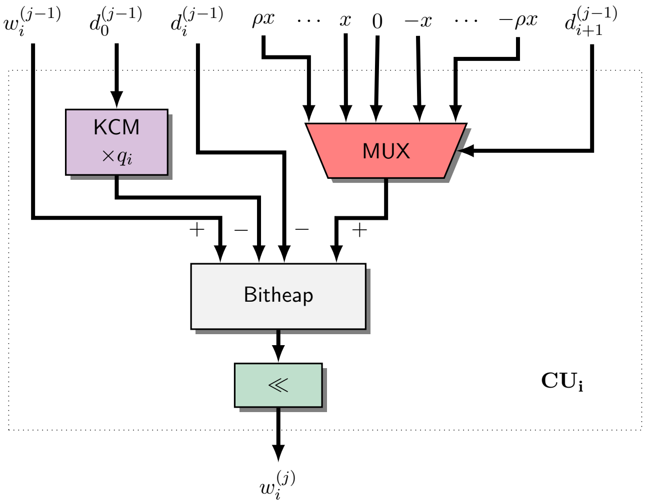
Our approach:

- based on [Brisebarre & Chevillard 2007, Brisebarre et al 2008]
- apply algorithms from Euclidean lattice theory
- lattice-based error $\simeq \mathbf{5.71 \cdot 10^{-11}} < 2^{-32}$ ☺

HW implementation of the E-method

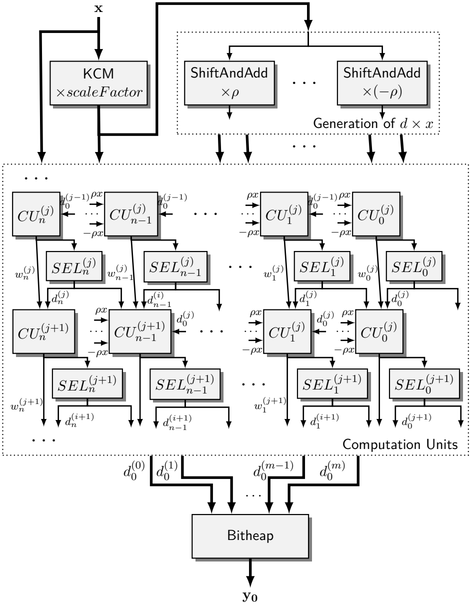
- generate circuit descriptions for FPGA devices
- flexibility in exploring different HW designs
- unrolled implementation of the method

Architecture of an iteration



$$w_i^{(j)} = r \cdot \left[w_i^{(j-1)} - q_i \cdot d_0^{(j-1)} - d_i^{(j-1)} + d_{i+1}^{(j-1)} \cdot x \right]$$

Architecture of an unrolled implementation



Optimizations

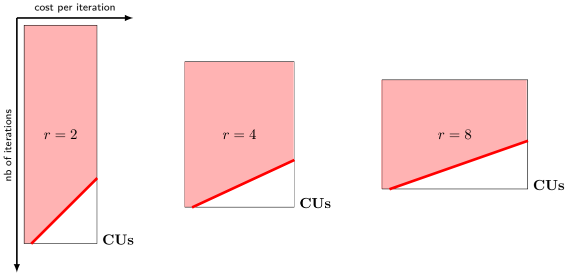
$$w_i^{(j)} = r \cdot \left[w_i^{(j-1)} - q_i \cdot d_0^{(j-1)} - d_i^{(j-1)} + d_{i+1}^{(j-1)} \cdot x \right]$$

- $q_i \cdot d_0^{(j-1)}$
  - multiplication by a constant
  - use the *KCM technique* [Chapman 1993]
- $d_{i+1}^{(j-1)} \cdot x$
  - pre-compute all products
  - reduced to a MUX
- $r \cdot$
  - shift left by one digit
  - on an FPGA, no hardware needed (wiring)
- **summation**
  - use a *bitheap* [Brunie et al 2013]
    - generalization of a compressor tree
  - optimal cost and execution time

Iteration-level optimizations

- **Iteration 0**:
  - initialization
  - store precomputed values
- **Iteration 1**:
  - precompute values
  - store in logic fabric
- **Iteration 2**:
  - precompute values (computations not involving x)
  - simpler iterations

Simplifying the last iterations



Some Results on a Xilinx Virtex6 device

Ex. 1: $f(x) = \sqrt{1 + (9x/2)^4}, x \in [0, 1/32]$

- target error 2^{-32}, approx. type: $(4, 4)$

Design	Approach	radix	Resources		Performance
			LUT	reg.	cycles@period(ns)
Ex. 1	Ours	2	7,880	0	1@94.3
			7,966	1,523	11@9.6
			7,299	2,689	17@5.7
			6,786	5,202	36@3.7
		4	4,871	0	1@57.9
			4,768	988	7@12.3
			4,600	1,583	11@6.9
			4,853	3,106	22@3.8
		8	4,210	0	1@44.4
			3,875*	0	1@62.2*
			5,307*	309	5@18.4*
			5,184*	499	8@10.4*
			4,707*	1,027	15@5.8*
	FloPoCo	–	994	0	1@29.5
			1,032	138	7@6.7
			1,147	335	19@5.3

More results

Ex. 2: $f(x) = \exp(2x)$, $x \in [0, 7/128]$

- target error 2^{-32}, approx. types: $(3,3), (4,4), (5,0)$.

Design	Approach	radix	Resources		Performance
			LUT	reg.	cycles@period(ns)
Ex. 2	Ours	2	6,820	0	1@88.5
		4	6,356	0	1@68.0
		8	5,042	0	1@39.0
	FloPoCo	–	3,024	0	1@41.1

Even more results

Ex. 3: $f(x) = \log_2\left(1 + 2^{-16x}\right), x \in [0, 1/16]$, approx. types: $(5,5), (5,0)$.
Ex. 4: $f(x) = \mathrm{erf}(x), x \in [0, 1/32]$, approx. types: $(4,4), (5,0)$.
Ex. 5: $f(x) = J_0(2x - 1/16), x \in [0, 1/16]$, approx. types: $(4,4), (6,0)$.

Design	Approach	radix	Resources		Performance	Target error
			LUT	reg.	cycles@period(ns)	
Ex. 3	Ours	2	2,944	0	1@67.0	2^{-24}
		4	2,742	0	1@35.1	
		8	2,582	0	1@33.1	
		16	2,856	0	1@31.2	
			1,565*	0	1@29.0*	
	FloPoCo	–	3,622	0	1@55.7	
Ex. 4	Ours	2	19,564	0	1@139.6	2^{-48}
		4	23,052	0	1@92.5	
			21,179*	0	1@131.5*	
		8	15,388*	0	1@250.7*	
		16	12,878*	0	1@76.9*	
		32	3,909*	0	1@86.7*	
	FloPoCo	–	20,494	0	1@139.9	
Ex. 5	Ours	2	19,423	0	1@368.1	2^{-48}
		4	13,642	0	1@70.3	
		8	18,653	0	1@58.6	
	FloPoCo	–	–	–	–	

Our work

- efficient methods for (quasi)optimal E-fraction approximation
- FPGA-optimized implementation of the E-method
  - efficient HW implementation of rational functions
  - customizable pipelined design $\rightarrow$ high throughput
- automatic open source tool for function evaluation
  - written in C++
  - available at: https://github.com/sfilip/emethod

Polynomial or rational function?

- depends on the problem!

Thank you!

Scan me!