On various ways to split a floating-point number

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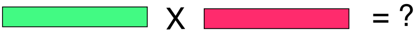
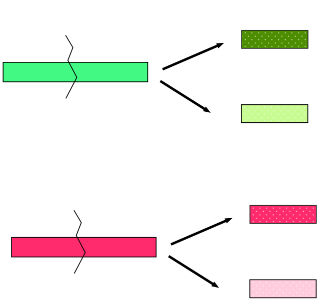
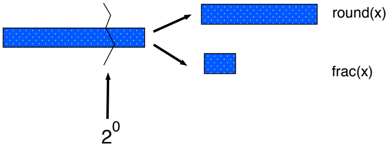
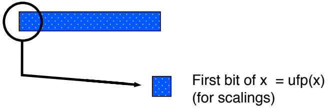
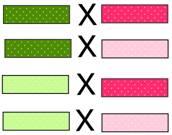
Splitting a floating-point number









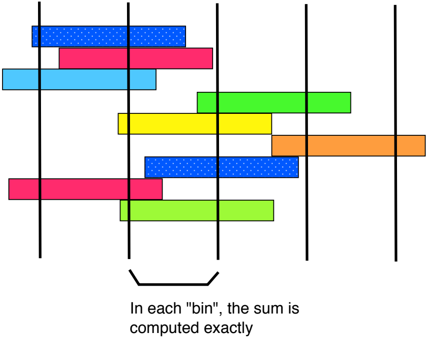


All products are computed exactly with one FP multiplication (Dekker product)

$$\sqrt{a^2 + b^2} \to 2^k \sqrt{\left(\frac{a}{2^k}\right)^2 + \left(\frac{b}{2^k}\right)^2}$$

Dekker product (1971)

Splitting a floating-point number



- Matlab program in a paper by Zielke and Drygalla (2003),
- analysed and improved by Rump, Ogita, and Oishi (2008),
- reproducible summation, by Demmel & Nguyen.

- absolute splittings (e.g., $\lfloor x \rfloor$), vs relative splittings (e.g., most significant bits, splitting of the significands for multiplication);
- no bit manipulations of the binary representations (would result in less portable programs) $\rightarrow$ only FP operations.

Notation and preliminary definitions

- IEEE-754 compliant FP arithmetic with radix β, precision p, and extremal exponents $e_{\min}$ and $e_{\max}$;
- $\mathbb{F}$ = set of FP numbers. $x \in \mathbb{F}$ can be written

$$x = \left(\frac{M}{\beta^{p-1}}\right) \cdot \beta^{e},$$

  M, $e \in \mathbb{Z}$, with $|M| < \beta^{p}$ and $e_{\min} \leqslant e \leqslant e_{\max}$, and $|M|$ maximum under these constraints;
- significand of x: $M \cdot \beta^{-p+1}$;
- RN = rounding to nearest with some given tie-breaking rule (assumed to be either "to even" or "to away", as in IEEE 754-2008);

Notation and preliminary definitions

Definition 1 (classical ulp)

The unit in the last place of $t \in \mathbb{R}$ is

$$\mathrm{ulp}(t) = \begin{cases} \beta^{\lfloor \log_\beta |t| \rfloor - p + 1} & \text{if } |t| \geqslant \beta^{e_{\min}}, \\ \beta^{e_{\min} - p + 1} & \text{otherwise.} \end{cases}$$

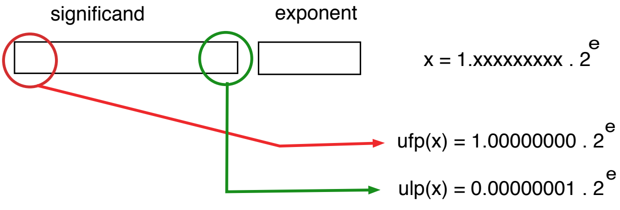
Definition 2 (ufp)

The *unit in the first place* of $t \in \mathbb{R}$ is

$$\mathrm{ufp}(t) = \begin{cases} \beta^{\lfloor \log_\beta |t| \rfloor} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

(introduced by Rump, Ogita and Oishi in 2007)

Notation and preliminary definitions



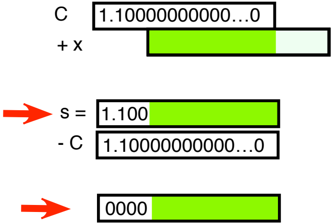
Guiding thread of the talk: *catastrophic cancellation is your friend.*

Absolute splittings: 1. nearest integer

Uses a constant C. Same operations as Fast2Sum, yet different assumptions.

Algorithm 1

Require: $C, x \in \mathbb{F}$
$s \leftarrow \mathrm{RN}(C + x)$
$x_h \leftarrow \mathrm{RN}(s - C)$
$x_\ell \leftarrow \mathrm{RN}(x - x_h)$ {optional}
return x_h {or (x_h, x_ℓ)}



First occurrence we found: Hecker (1996) in radix 2 with $C = 2^{p-1}$ or $C = 2^{p-1} + 2^{p-2}$. Use of latter constant referred to as the 1.5 trick.

Theorem 3

Assume C integer with $\beta^{p-1} \leqslant C \leqslant \beta^p$. If $\beta^{p-1} - C \leqslant x \leqslant \beta^p - C$, then x_h is an integer such that $|x - x_h| \leqslant 1/2$. Furthermore, $x = x_h + x_\ell$.

Absolute splittings: 2. floor function

An interesting question is to compute $\lfloor x \rfloor$, or more generally $\lfloor x/\beta^k \rfloor$.

Algorithm 2

Require: $x \in \mathbb{F}$

$y \leftarrow \mathrm{RN}(x - 0.5)$

$C \leftarrow \mathrm{RN}(\beta^p - x)$

$s \leftarrow \mathrm{RN}(C + y)$

$x_h \leftarrow \mathrm{RN}(s - C)$

return x_h

Theorem 4

Assume β is even, $x \in \mathbb{F}$, $0 \leqslant x \leqslant \beta^{p-1}$. Then Algorithm 2 returns $x_h = \lfloor x \rfloor$.

Relative splittings

- expressing a precision-p FP number x as the exact sum of a $(p-s)$-digit number x_h and an s-digit number x_ℓ;
- first use with $s = \lfloor p/2 \rfloor$ (Dekker product, 1971)
- another use: $s = p-1 \rightarrow x_h$ is a power of β giving the order of magnitude of x. Two uses:
  - evaluate $\mathrm{ulp}(x)$ or $\mathrm{ufp}(x)$. Useful functions in the error analysis of FP algorithms;

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    - power of β close to $|x|$: for scaling x, such a weaker condition suffices, and can be satisfied using fewer operations.

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  - power of β close to $|x|$: for scaling x, such a weaker condition suffices, and can be satisfied using fewer operations.

    $\rightarrow$ approximate information

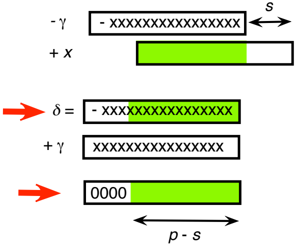
Veltkamp splitting

$x \in \mathbb{F}$ and $s < p \rightarrow$ two FP numbers x_h and x_ℓ s.t. $x = x_h + x_\ell$, with the significand of x_h fitting in $p - s$ digits, and the one of x_ℓ in s digits ($s - 1$ when $\beta = 2$ and $s \geqslant 2$).

Algorithm 3 Veltkamp's splitting.

Require: $C = \beta^s + 1$ and x in $\mathbb{F}$
$\gamma \leftarrow \mathrm{RN}(Cx)$
$\delta \leftarrow \mathrm{RN}(x - \gamma)$
$x_h \leftarrow \mathrm{RN}(\gamma + \delta)$
$x_\ell \leftarrow \mathrm{RN}(x - x_h)$
return (x_h, x_ℓ)

Remember: *catastrophic cancellation is your friend!*



- Dekker (1971): radix 2 analysis, implicitly assuming no overflow;
- extended to any radix β by Linnainmaa (1981);
- works correctly even in the presence of underflows;
- Boldo (2006): Cx does not overflow $\Rightarrow$ no other operation overflows.

Veltkamp splitting: FMA variant

If an FMA instruction is available, we suggest the following variant, that requires fewer operations.

Algorithm 4 FMA-based relative splitting.

Require: $C = \beta^s + 1$ and x in $\mathbb{F}$
$\gamma \leftarrow \mathrm{RN}(Cx)$
$x_h \leftarrow \mathrm{RN}(\gamma - \beta^s x)$
$x_\ell \leftarrow \mathrm{RN}(Cx - \gamma)$
return (x_h, x_ℓ)

Remarks

- x_ℓ obtained in parallel with x_h
- even without an FMA, γ and $\beta^s x$ can be computed in parallel,
- the bounds on the numbers of digits of x_h and x_ℓ given by Theorem 5 can be attained.

Theorem 5

Let $x \in \mathbb{F}$ and $s \in \mathbb{Z}$ s.t. $1 \leqslant s < p$. Barring underflow and overflow, Algorithm 4 computes $x_h, x_\ell \in \mathbb{F}$ s.t. $x = x_h + x_\ell$. If $\beta = 2$, the significands of x_h and x_ℓ have at most $p - s$ and s bits, respectively. If $\beta > 2$ then they have at most $p - s + 1$ and $s + 1$ digits, respectively.

Extracting a single bit (radix 2)

- computing ufp(x) or ulp(x), or scaling x;
- Veltkamp's splitting (Algorithm 3) to x with $s = p - 1$: the resulting x_h has a 1-bit significand and it is nearest x in precision $p - s = 1$.
- For computing $\text{sign}(x) \cdot \text{ufp}(x)$, we can use the following algorithm, introduced by Rump (2009).

Algorithm 5

Require: $\beta = 2$, $\varphi = 2^{p-1}+1$, $\psi = 1 - 2^{-p}$, and $x \in \mathbb{F}$
$q \leftarrow \text{RN}(\varphi x)$
$r \leftarrow \text{RN}(\psi q)$
$\delta \leftarrow \text{RN}(q - r)$
return δ

Very rough explanation:

- $q \approx 2^{p-1}x + x$
- $r \approx 2^{p-1}x$

$\rightarrow$ $q - r \approx x$ but in the massive cancellation we loose all bits but the most significant.

Extracting a single bit (radix 2)

These solutions raise the following issues.

- If $|x|$ is large, then an overflow can occur in the first line of both Algorithms 3 and 5.
- To avoid overflow in Algorithm 5: scale it by replacing φ by $\frac{1}{2} + 2^{-p}$ and returning $2^p\delta$ at the end. However, this variant will not work for subnormal x.

$\rightarrow$ to use Algorithm 5, we somehow need to check the order of magnitude of x.

- If we are only interested in scaling x, then requiring the exact value of $\text{ufp}(x)$ is overkill: one can get a power of 2 “close” to x with a cheaper algorithm.

Extracting a single bit (radix 2)

Algorithm 6 $\text{sign}(x) \cdot \text{ulp}(x)$ for radix 2 and $|x| > 2^{e_{\min}}$.

Require: $\beta = 2$, $\psi = 1 - 2^{-p}$, and $x \in \mathbb{F}$
  $a \leftarrow \text{RN}(\psi x)$
  $\delta \leftarrow \text{RN}(x - a)$
  return δ

Theorem 6

If $|x| > 2^{e_{\min}}$, then Algorithm 6 returns

$$\text{sign}(x) \cdot \begin{cases} \frac{1}{2}\text{ulp}(x) & \textit{if } |x| \textit{ is a power of } 2, \\ \text{ulp}(x) & \textit{otherwise.} \end{cases}$$

Similar algorithm for $\text{ufp}(x)$, under the condition $|x| < 2^{e_{\max}-p+1}$.

Underflow-safe and almost overflow-free scaling

- $\beta = 2$, $p \geqslant 4$;
- RN breaks ties "to even" or "to away";

Given a nonzero FP number x, compute a scaling factor δ s.t.:

- $|x|/\delta$ is much above the underflow threshold and much below the overflow threshold (so that, for example, we can safely square it);
- δ is an integer power of 2 ($\rightarrow$ no rounding errors when multiplying or dividing by it).

Algorithms proposed just before: simple, but underflow or overflow can occur for many inputs x.

Underflow-safe and almost overflow-free scaling

Following algorithm: underflow-safe and *almost* overflow-free in the sense that only the two extreme values $x = \pm(2 - 2^{1-p}) \cdot 2^{e_{\max}}$ must be excluded.

Algorithm 7

Require: $\beta = 2$, $\Phi = 2^{-p} + 2^{-2p+1}$, $\eta = 2^{e_{\min}-p+1}$, and $x \in \mathbb{F}$

$y \leftarrow |x|$

$e \leftarrow \mathrm{RN}(\Phi y + \eta)$ {or $e \leftarrow \mathrm{RN}(\mathrm{RN}(\Phi y) + \eta)$ without FMA}

$y_{\sup} \leftarrow \mathrm{RN}(y + e)$

$\delta \leftarrow \mathrm{RN}(y_{\sup} - y)$

return δ

Underflow-safe and almost overflow-free scaling

First 3 lines of Algorithm 7: algorithm due to Rump, Zimmermann, Boldo and Melquiond, that computes the FP successor of $x \notin [2^{e_{\min}}, 2^{e_{\min}+2}]$. We have,

Theorem 7

For $x \in \mathbb{F}$ with $|x| \neq (2 - 2^{1-p}) \cdot 2^{e_{\max}}$, the value δ returned by Algorithm 7 satisfies:

- *if* RN *is with "ties to even" then δ is a power of 2;*
- *if* RN *is with "ties to away" then δ is a power of 2, unless $|x| = 2^{e_{\min}+1} - 2^{e_{\min}-p+1}$, in which case it equals $3 \cdot 2^{e_{\min}-p+1}$;*
- *if $x \neq 0$, then*

$$1 \leqslant \left|\frac{x}{\delta}\right| \leqslant 2^p - 1.$$

$\rightarrow$ makes δ a good candidate for scaling x;

$\rightarrow$ in the paper: application to $\sqrt{a^2 + b^2}$.

Experimental results

Although we considered floating-point operations only, we can compare with bit-manipulations.

The C programs we used are publicly available (see proceedings).

Experimental setup: Intel i5-4590 processor, Debian testing, GCC 7.3.0 with -O3 optimization level, FPU control set to rounding to double.

Computation of `round` or `floor`:

	`round`	`floor`
Algorithms 1 and 2	**0.106s**	**0.173s**
Bit manipulation	0.302s	0.203s
GNU libm `rint` and `floor`	0.146s	0.209s

Note: Algorithms 1 and 2 require $|x| \leqslant 2^{51}$ and $0 \leqslant x \leqslant 2^{52}$ respectively.

Relative splitting of a double-precision number

Splitting into x_h and x_ℓ:

	x_h	x_ℓ	time
Algorithm 3	26 bits	26 bits	0.108s
Algorithm 4	26 bits	27 bits	0.106s
Algorithm 4 with FMA	26 bits	27 bits	0.108s
Bit manipulation	26 bits	27 bits	**0.095s**

Algorithms 3 and 4 assume no intermediate overflow or underflow.

Conclusion

- systematic review of splitting algorithms
- found some new algorithms, in particular with FMA
- many applications for absolute and relative splitting
- in their application range, these algorithms are competitive with (less-portable) bit-manipulation algorithms

Motivation

Question of Pierrick Gaudry (Caramba team, Nancy, France):

Multiple-precision integer arithmetic in Javascript.

Javascript has only a 32-bit integer type, but 53-bit doubles!

Storing 16-bit integers in a double precision register, we can accumulate up to 2^{21} products of 32 bits, and then have to perform `floor(x/65536.0)` to normalize.

The Javascript code `Math.Floor(x/65536.0)` is slow on old internet browsers (Internet Explorer version 7 or 8)!

The Javascript standard says it is IEEE754, with always round to nearest, ties to even.

Pierrick Gaudry then opened the "Handbook of Floating-Point Arithmetic"...

First algorithm (designed by Pierrick Gaudry):

Assume $0 \leqslant x < 2^{36}$ and x is an integer

We can compute `floor(x)` as follows:

Let $C = 2^{36} - 2^{-1} + 2^{-17}$.

$s \leftarrow \mathrm{RN}(C + x)$

Return $\mathrm{RN}(s - C)$

Question: can we get rid of the condition "x integer"?